PHILLIPS-LOOMIS MATHEMATICAL SERIES

ELEMENTS OF GEOMETRY

BY

ANDREW W. PHILLIPS, PH.D.

AND

IRVING FISHER, PH.D.

PROFESSORS IN YALE UNIVERSITY

ABRIDGED EDITION

NEW YORK AND LONDON

HARPER & BROTHERS PUBLISHERS

1897

THE PHILLIPS-LOOMIS MATHEMATICAL SERIES.

ELEMENTS OF GEOMETRY. By ANDREW W. PHILLIPS, Ph.D., and IRVING FISHER, Ph.D., Professors in Yale University. With an Appendix treating of Plane Curves and Plane Figures, Exercises in Plane and Solid Geometry, and Introduction to Modern Geometry, etc. Crown 8vo, Half Leather, $1 75. [By mail, $1 92.]

ABRIDGED EDITION. Crown 8vo, $1 25. [By mail, $1 40.]

PLANE GEOMETRY. By ANDREW W. PHILLIPS, Ph.D., and IRVING FISHER, Ph.D. Crown 8vo, Cloth, 80 cents. [By mail, 90 cents.]

KEY TO THE ELEMENTS OF GEOMETRY. (*In Press.*)

LOGARITHMS OF NUMBERS. Five-Figure Table to Accompany the "Elements of Geometry," by ANDREW W. PHILLIPS, Ph.D., and IRVING FISHER, Ph.D., Professors in Yale University. Crown 8vo, Cloth, 30 cents. [By mail, 35 cents.]

PUBLISHED BY HARPER & BROTHERS, NEW YORK.

PREFACE

THIS book is an abridgment of the authors' "Elements of Geometry," and has been made for those schools whose course of study in this subject does not admit of the use of a more extended work.

The following are some of its more important features:

The *Introduction* presents in the shortest possible compass the general outlines of the science to be studied, and leads at once to the actual study itself.

The *definitions* are distributed through the book as they are needed, instead of being grouped in long lists many pages in advance of the propositions to which they apply. An alphabetical index is added for easy reference.

The *constructions* in the Plane Geometry are also distributed, so that the student is taught how to make a figure at the same time that he is required to use it in demonstration.

In the Geometry of Space, the figures consist of half-tone engravings from the *photographs of actual models* recently constructed for use in the class-rooms of Yale University. By the side of these models are skeleton diagrams for the student to copy.

Extensive use has been made of *natural* and *symmetrical* methods of demonstration. Such methods are used for deducing the formula for the sum of the angles of a triangle, for the sum of the exterior and interior angles of a polygon, for parallel lines, for the theorems on regular polygons, and for similar figures.

Attention is also called to the methods employed in dealing with *proportion*, and to the rigorous treatment of the *theory of limits*.

The corollaries, and the exercises at the end of each book, will ordinarily give the student ample practice in "inventional geometry," while in exceptional cases the student is referred to the carefully selected collection of difficult exercises at the end of the "Elements of Geometry."

This abridgment has been made by our colleague, Mr. Wendell M. Strong, whose equipment has fitted him in a peculiar manner for the task.

ANDREW W. PHILLIPS.
IRVING FISHER.

YALE UNIVERSITY, *June*, 1897.

TABLE OF CONTENTS

INTRODUCTION

BOOK I

BOOK II

BOOK III

BOOK IV

BOOK IX

INDEXES

SPECIAL TERMS

An **axiom** is a truth assumed as self-evident.

A **theorem** is a truth which becomes evident by a train of reasoning called a **demonstration.**

A theorem consists of two parts, the *hypothesis*, that which is given, and the *conclusion*, that which is to be proved.

A **problem** is a question proposed which requires a solution.

A **proposition** is a general term for either a theorem or problem.

One theorem is the **converse** of another when the conclusion of the first is made the hypothesis of the second, and the hypothesis of the first is made the conclusion of the second.

The converse of a truth is not always true. Thus, "If a man is in New York City he is in New York State," is true; but the converse, "If a man is in New York State he is in New York City," is not necessarily true.

When one theorem is easily deduced from another the first is sometimes called a **corollary** of the second.

A theorem used merely to prepare the way for another theorem is sometimes called a **lemma.**

SYMBOLS AND ABBREVIATIONS

$+$ plus.	Cons.—Construction.
$-$ minus.	Cor.—Corollary.
$>$ is greater than.	Def.—Definition.
$<$ is less than.	Fig.—Figure.
$\times$ multiplied by.	Hyp.—Hypothesis.
$=$ equals.	Iden.—Identical.
$\asymp$ is equivalent to.	Q. E. D.—Quod erat demonstrandum.
Alt.-int.—Alternate interior.	Q. E. F.—Quod erat faciendum.
Ax.—Axiom.	Sup.-adj.—Supplementary adjacent.

GEOMETRY

INTRODUCTION

FUNDAMENTAL CONCEPTIONS

1. *Def.*—**Geometry** is the science of **space.**

2. Every one has a notion of space extending indefinitely in all directions. Every material body, as a rock, a tree, or a house, occupies a limited portion of space. The portion of space which a body occupies, considered separately from the matter of which it is composed, is a *geometrical solid.* The material body is a *physical solid.* Only geometrical solids are here considered, and they are called simply *solids.*

Def.—A **solid** is, then, a limited portion of **space.**

3. *Def.*—The boundaries of a solid are **surfaces** (that is, the surfaces separate it from the surrounding space).

A surface is no part of a solid.

4. *Def.*—The boundaries of a surface are **lines.**

A line is no part of a surface.

5. *Def.*—The boundaries (or ends) of a line are **points.**

A point is no part of a line.

6. The solid, surface, line, and point are the four fundamental conceptions of geometry. They may also be considered in the reverse order, thus:

(1.) A **point** has position but no magnitude.

(2.) If a point moves, it generates (traces) a **line**.
This motion gives to the line its only magnitude, *length*.

(3.) If a line moves (not along itself), it generates a **surface**.
This motion gives to the surface, besides length, *breadth*.

(4.) If a surface moves (not along itself), it generates a **solid**.
This motion gives to the solid, besides length and breadth, *thickness*.

Def.—A **figure** is any combination of points, lines, surfaces, or solids.

7. *Def.*—A **straight line** is a line which is the shortest path between any two of its points.

8. *Def.*—A **plane surface** (or simply a **plane**) is a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface.

9. *Def.*—Two straight lines are **parallel** which lie in the same plane and never meet, however far produced.

GEOMETRIC AXIOMS

10. All the truths of geometry rest upon three fundamental axioms, viz.:

(*a.*) **Straight line axiom.**—Through every two points in space there is one and only one straight line.

This is sometimes expressed as follows: Two points *determine* a straight line.

(*b.*) **Parallel axiom.**—Through a given point there is one and only one straight line parallel to a given straight line.

(*c.*) **Superposition axiom.**—Any figure in a plane may be freely moved about in that plane without change of size or shape. Likewise, any figure in space may be freely moved about in space without change of size or shape.

GENERAL AXIOMS

11. In reasoning from one geometric truth to another the following general axioms are also employed, viz.:

(1.) Things equal to the same thing are equal to each other.

(2.) If equals be added to equals, the wholes are equal.

(3.) If equals be taken from equals, the remainders are equal.

(4.) If equals be added to unequals, the wholes are unequal in the same order.

(5.) If equals be taken from unequals, the remainders are unequal in the same order.

(6.) If unequals be taken from equals, the remainders are unequal in the opposite order.

(7.) If equals be multiplied by equals, the products are equal; and if unequals be multiplied by equals, the products are unequal in the same order.

(8.) If equals be divided by equals, the quotients are equal; and if unequals be divided by equals, the quotients are unequal in the same order.

(9.) If unequals be added to unequals, the lesser to the lesser and the greater to the greater, the wholes will be unequal in the same order.

(10.) The whole is greater than any of its parts.

(11.) The whole is equal to the sum of all its parts.

(12.) If of two unequal quantities the lesser increases (continuously and indefinitely) while the greater decreases; they must become equal once and but once.

(13.) If of three quantities the first is greater than the second and the second greater than the third, then the first is greater than the third.

12. *Def.*—**Plane Geometry** treats of figures in the same plane.

13. *Def.*—**Solid Geometry**, or the **geometry of space**, treats of figures not wholly in the same plane.

PLANE GEOMETRY

BOOK I

FIGURES FORMED BY STRAIGHT LINES

14. *Defs.*—An **angle** is a figure formed by two straight lines diverging from the same point.

This point is the *vertex* of the angle, and the lines are its *sides*.

A clear notion of an angle may be obtained by observing the hands of a clock, which form a continually varying angle.

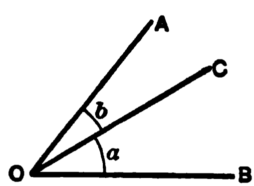


FIG. 1

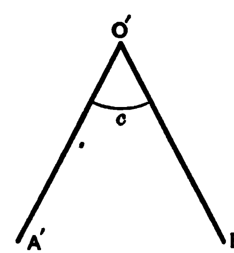


FIG. 2

We may designate an angle by a letter placed within as a and b in Fig. 1, and c in Fig. 2.

Three letters may be used, viz.: one letter on each of its sides, together with one at the vertex, which must be written between the other two, as AOC, BOC, and AOB in Fig. 1, and $A'O'B'$ in Fig. 2.

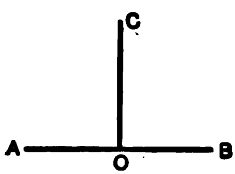
If there is but one angle at a point, it may be denoted by a single letter at that point, as O' in Fig. 2.

Angles with a common vertex and side, as a and b, are said to be **adjacent.**

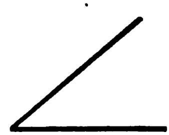
15. *Def.*—Two angles are **equal** if they can be made to coincide. Also, in general, any two figures are equal which can be made to coincide.

Thus, suppose we place the angle AOB on the angle $A'O'B'$ so that O shall fall at O', and the side OA along $O'A'$; then, if the side OB also falls along $O'B'$, the angles are equal, *whatever may be the length of each of their sides.*

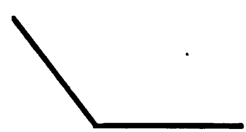
16. *Def.*—When one straight line is drawn from a point in another, so that the two adjacent angles are equal, each of these angles is a **right angle**, and the lines are **perpendicular.**



RIGHT ANGLES



ACUTE ANGLE



OBTUSE ANGLE

Thus, if the angles AOC and COB are equal, they are right angles, and CO is perpendicular to AB.

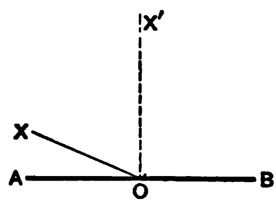
When a straight line is perpendicular to another straight line, its point of intersection with the second line is called the **foot of the perpendicular.**

17. *Def.*—An **acute** angle is an angle less than a right angle; an **obtuse** angle, greater.

The term **oblique** angle may be applied to any angle which is not a right angle.

PROPOSITION I. THEOREM

18. *From a point in a straight line one perpendicular, and only one, can be drawn (on the same side of the given straight line).*



GIVEN a straight line, AB, and any point, O, upon it.

TO PROVE—from O one, and only one, perpendicular can be drawn to AB (on the same side of AB).

Suppose a straight line OX to revolve about O. Ax. *c*

In every one of its successive positions it forms two different angles with the line AB, viz.: XOA and XOB.

As it revolves from the position OA around to the position OB the lesser angle, XOA, will continuously increase, and the other, XOB, will continuously decrease.

There must, therefore, be one and only one position of OX, as OX' where the angles become equal. Ax. 12

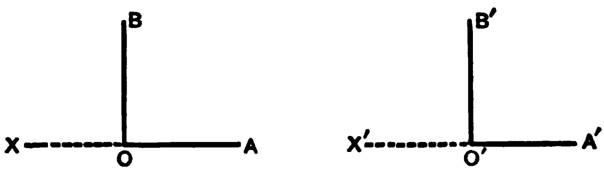
[If, of two unequal quantities, the lesser increases, etc.]

That is, there must be one and only one perpendicular to AB at O. Q. E. D.

Question.—The above proposition applies to the plane of the diagram. Could you draw any other lines perpendicular to AB at O *out* of the plane of the page?

PROPOSITION II. THEOREM

19. *All right angles are equal.*



GIVEN any two right angles AOB and $A'O'B'$.

TO PROVE they are equal.

Apply $A'O'B'$ to AOB so that the vertex O' shall fall on O, and so that A', any point in one side of $A'O'B'$, shall fall on some point in OA or OA produced.

Then the line $O'A'$ will coincide with OA, even if both be produced indefinitely. Ax. *a*

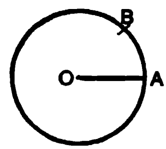
[Two points determine a straight line.]

If $O'B'$ should not fall along OB, there would be two lines, $O'B'$ and OB, perpendicular to the same line from the same point, which is impossible. § 18

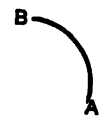
[From a point in a straight line, one perpendicular, and only one can be drawn.]

Therefore $O'B'$ must fall along OB—that is, the angles $A'O'B'$ and AOB coincide and are equal. Q. E. D.

20. *Defs.*—A **circle** is a figure bounded by a line all points of which are equally distant from a point within called the **centre.**



CIRCLE



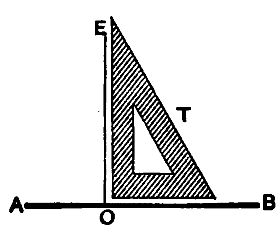
ARC

The bounding line is called the **circumference.**

Any portion of the circumference is called an **arc.**

Any one of the equal lines from the centre to the circumference (as OA) is called a **radius.**

21. CONSTRUCTION. *To draw a perpendicular from a straight line AB at some point in it, as O.*

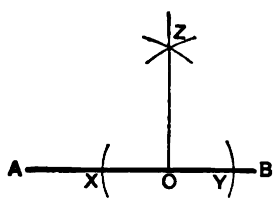


FIRST METHOD

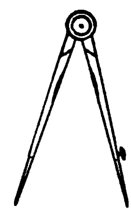
First method.—Place a right-angled ruler T with the vertex of its right angle at O and one of its edges along AB. Draw OE along its other edge. OE will be the required line, for, first, it is drawn through O, second, it is drawn perpendicular to AB.

The student should observe that it is impossible to construct an absolutely accurate diagram, for no ruler is absolutely accurate nor can it be applied with absolute accuracy. Moreover the dots and marks formed by a pencil, however well sharpened, are not absolute points and lines, for the dots have *some* magnitude, and the marks *some* breadth. Diagrams only *approximate* the ideal points and lines intended.

If, however, the practical means employed *could be made perfect*, the resulting construction *would be* absolutely exact. Hence we may say of the preceding construction, the *method* is perfect, though the *means* can never be. This method is largely used by draughtsmen and carpenters.



SECOND METHOD



COMPASSES

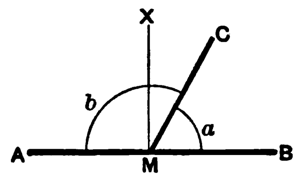
Second method (with straight ruler and compasses).—Take O as a centre, and with any convenient radius describe with the compasses two arcs cutting AB at X and Y. Then with X and Y as centres, with a somewhat longer radius describe two arcs cutting each other at Z. Draw the straight line OZ by means of the ruler. OZ will be the perpendicular required.

[The correctness of the second method can be proved after reaching § 88.]

Of the two methods above described, the first has the advantage of quickness, but it assumes that the ruler is really made with a right angle, that is, it assumes that some one has already constructed a right angle and all we do is to copy it. The second method is free from this assumption, though, in both methods, it is assumed that the ruler is made with a straight edge, that is, that some one has already constructed a straight line. The first way of constructing a straight line was by stretching a string, a method still used by carpenters. In fact the word "straight" originally meant "stretched." The ancient Egyptians used this method, and even invented a way of making a right angle by stretching a cord. (See foot-note to § 305.)

PROPOSITION III. THEOREM

22. *The two angles which one straight line makes with another, upon one side of it, are together equal to two right angles.*



GIVEN—the straight line CM meeting the straight line AB at M and forming the angles a and b.

TO PROVE $a+b=2$ right angles.

Suppose MX drawn perpendicular to AB. § 18

[From a point in a straight line one perpendicular can be drawn.]

Then $BMX+XMA=2$ right angles. § 16

We may substitute for BMX its equal, $a+CMX$. Ax. 11

[A whole is equal to the sum of its parts.]

This gives $a+CMX+XMA=2$ right angles.

We may now substitute for $CMX+XMA$ the angle b.

[Same axiom.]

This gives $a+b=2$ right angles. Q. E. D.

23. *Defs.*—Two angles whose sum is equal to a right angle, are **complementary** angles.

Two angles whose sum is two right angles, are **supplementary** angles.

The two angles which one straight line makes with another on one side of it (as a and b), are **supplementary-adjacent** angles.

24. COR. I. *If one of the angles formed by the intersection of two straight lines is a right angle, the others are right angles.* (Fig. 1.)

Hint.—Apply Proposition III.

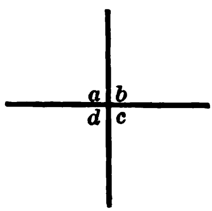


FIG. 1

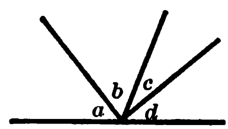


FIG. 2

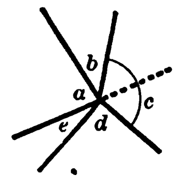


FIG. 3

25. COR. II. *If of two intersecting straight lines one is perpendicular to the other, then the second is also perpendicular to the first.*

Hint.—Apply Corollary I.

26. In COROLLARIES the proof is left, wholly or in part, to the student. Practice will give him the power of carefully stating and separating the steps and *finding for each a satisfactory reason.*

27. COR. III. *The sum of all the angles about a point on one side of a straight line equals two right angles.* (Fig. 2.)

Hint.—Group the angles into two angles and apply Proposition III.

28. COR. IV. *The sum of all the angles about a point equals four right angles.* (Fig. 3.)

Hint.—Prolong one of the lines through the vertex, separating the opposite angle *c* into two angles, and apply Corollary III.

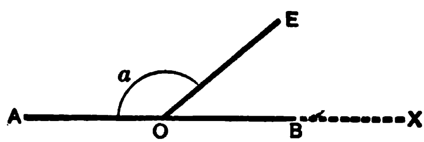
Question.—If, of three angles around a point, two are each one and a third right angles, how much is the third angle?

Question.—If six angles about a point are all equal, how large is each angle?

PROPOSITION IV. THEOREM

29. *If two adjacent angles are together equal to two right angles, their exterior sides are in the same straight line.*

[The converse of Proposition III.]



GIVEN $a+EOB=2$ right angles.

TO PROVE AO and OB form one straight line.

Let OX be the prolongation of AO.

$a+EOB=2$ right angles. Hyp.

$a+EOX=2$ right angles. § 22

[Being sup.-adj.]

Hence $a+EOB=a+EOX$. Ax. 1

Subtracting a, $EOB=EOX$. Ax. 3

Hence OB must coincide with OX.

Otherwise one of the angles (EOB and EOX) would include the other, and they could *not* be equal. Ax. 10

Therefore OB lies in the same straight line with OA.

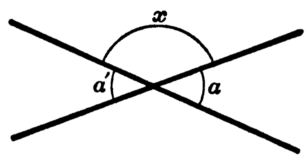
Q. E. D.

Question.—If two angles are supplementary-adjacent, and their difference is one right angle, how large is each?

Question.—The angles on the same side of a straight line are three in number. The greatest is three times the least, and the remaining one is twice the least. How large is each? In how many ways can they be arranged on the straight line?

PROPOSITION V. THEOREM

30. *If two straight lines intersect, the opposite (or vertical) angles are equal.*



GIVEN—two intersecting straight lines forming the opposite angles a and a'.

TO PROVE $a = a'$.

	$a + x = 2$ right angles.	§ 22
	$a' + x = 2$ right angles.	§ 22
	[Being, in each case, sup.-adj.]	
Therefore	$a + x = a' + x$.	Ax. 1
Subtracting x,	$a = a'$.	Ax. 3
		Q. E. D.

PARALLEL LINES AND SYMMETRICAL FIGURES

31. *Def.*—Two straight lines are **parallel** which lie in the same plane, but never meet, however far produced.

PARALLEL LINES

32. *Def.*—Two figures are **symmetrical with respect to a straight line** called an **axis of symmetry,** when, if one of them be folded over on that line as an axis, it will coincide with the other. (Fig. 1.)

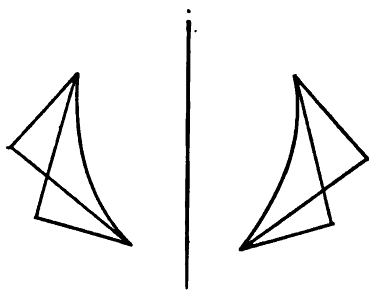


FIG. 1

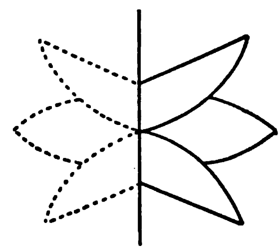
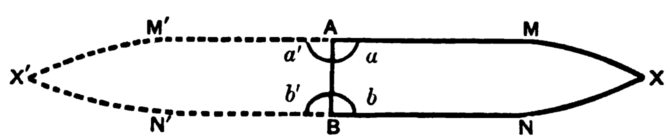


FIG. 2

A clear notion of this kind of symmetry may be obtained by drawing any figure in ink, and before the ink has dried folding the paper on to itself over a crease. The original figure and the resulting impression are symmetrical with respect to the crease as an axis. (Fig. 2.)

PROPOSITION VI. THEOREM

33. *Two straight lines perpendicular to the same straight line are parallel.*



GIVEN AM and BN perpendicular to AB.

TO PROVE AM and BN parallel.

If AM and BN should meet, either at the right or left, as at X, fold the figure AXB about AB as an axis to form the symmetrical impression $AX'B$, the right angles a and b forming the impressions a' and b' respectively.

Then AM and AM' form one and the same straight line,
and BN and BN' form one and the same straight line. § 29

[If two adjacent angles (as a' and a) are together equal to two right angles, their exterior sides are in the same straight line.]

Hence we would have two straight lines through X and X', which is absurd. Ax. a

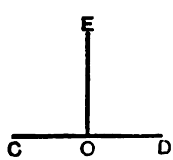
[Two points determine a straight line.]

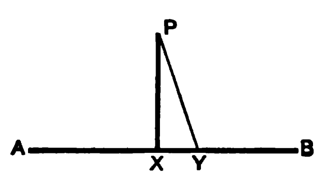
Therefore AM and BN cannot meet, and, as they lie in the same plane, they must be parallel. § 31

Q. E. D.

Question.—Will the preceding proposition still be true if the lines are not all confined to one plane?

34. COR. *Through a given point P without the line one and only one perpendicular can be drawn to a given straight line, AB.*

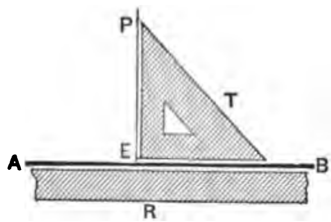




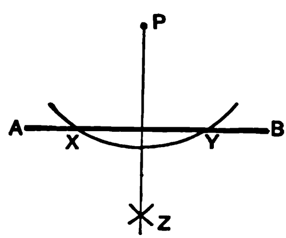
OUTLINE PROOF: From O in another line CD erect a perpendicular OE. (By what authority?) Superpose CD upon AB, and move it along AB until OE contains P. (What axiom applies?)

Second, suppose two were possible, as PX and PY, and show that this would contradict Proposition VI.

35. CONSTRUCTION. *To drop a perpendicular to a straight line AB from a point P without the line.*



First method.—Apply a straight edge of a ruler R to the straight line AB. Place one side of a right-angled ruler T upon the ruler R, making another side perpendicular to AB. Then slide T along AB until the perpendicular edge contains P. Draw PE along that edge. PE is the perpendicular required, for it is drawn through P and is perpendicular to AB.



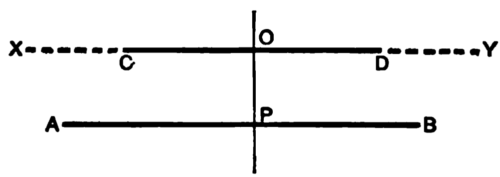
Second method.—From P as a centre with a convenient radius describe an arc cutting AB at X and Y. Then with X and Y in turn as centres describe arcs with equal radii intersecting at Z. Join PZ. This will be the required perpendicular.

[This can be proved correct after reaching § 104.]

PROPOSITION VII. THEOREM

36. *If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.*

[Converse of Proposition VI.]



GIVEN—CD and AB parallel, and PO perpendicular to AB.

TO PROVE PO perpendicular to CD.

Suppose XY to be drawn through O perpendicular to OP.
Then XY is parallel to AB. § 33

[Two straight lines perpendicular to the same straight line are parallel.]

But CD is parallel to AB. Hyp.
Hence CD must coincide with XY. Ax. *b*

[Through any point there is one and only one straight line parallel to a given straight line.]

That is CD must be perpendicular to PO,
and OP is perpendicular to CD. § 25

Q. E. D.

37. CONSTRUCTION. *To draw a straight line through a given point C parallel to a given straight line AB.*

First method (Fig. 1).—Place a right-angled ruler in the position T, making one edge about the right angle coincident with AB, and

along the other edge place a ruler R. Then hold the ruler R firmly against the paper. Slide T to the position T' till its edge reaches C. Draw CX. It is the parallel required. (Why?)

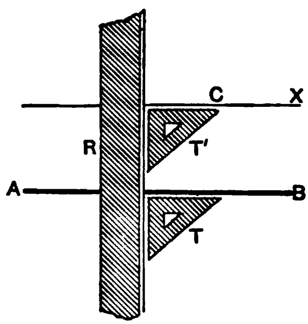


FIG. 1

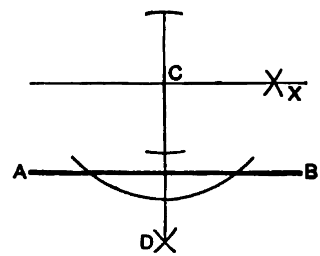
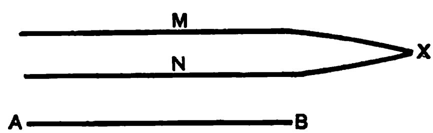


FIG. 2

Second method (Fig. 2).—From C draw CD perpendicular to AB. § 35
At C draw CX perpendicular to CD. § 21
Then CX is the required parallel to AB. (Why?)

PROPOSITION VIII. THEOREM

38. *If two straight lines are parallel to a third straight line, they are parallel to each other.*



GIVEN M and N each parallel to AB.

TO PROVE M and N parallel to each other.

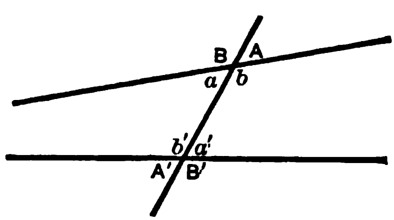
If M and N should meet, as at X, we would have two parallels to AB through the same point X, which is absurd. Ax. *b*

[Through one point there is one and only one straight line parallel to a given straight line.]

Therefore M and N cannot meet, and, lying in the same plane, must be parallel. § 31

Q. E. D.

39. *Defs.*—When two straight lines are cut by a third straight line, of the eight angles formed—



a, b, a', b', are **interior** angles.

A, B, A', B', are **exterior** angles.

a and a', or b and b', are **alternate-interior** angles.

A and A', or B and B', are **alternate-exterior** angles.

A and a', b and B', B and b', or a and A', are **corresponding** angles.

Question.—Of the eight angles, which are always equal, and why?

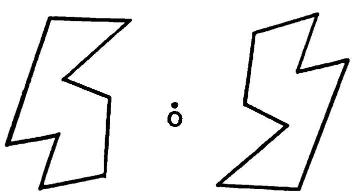
Question.—If $A = A'$, what other angles are also equal to A, and why? Are the remaining angles all equal, and if so, why?

Question.—If $A = A'$ and also $A = B$, what angles are equal, and why?

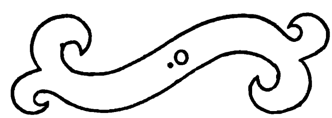
40. *Defs.*—Two figures are **symmetrical with respect to a point** called the **centre of symmetry** when, if one of them is revolved half way round on this point as a pivot, it will coincide with the other.

A single figure is said to be symmetrical with respect to a point called the centre of symmetry if, when the figure is turned half way round on this point as a pivot, each portion of the figure will take the position previously occupied by another part.

[A figure is said to be turned half way round a point when a line through the point turns through two right angles.]



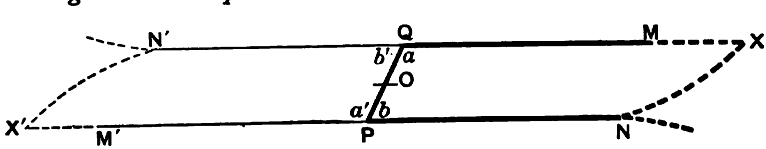
TWO FIGURES SYMMETRICAL WITH RESPECT TO O



A SINGLE FIGURE SYMMETRICAL WITH RESPECT TO O

PROPOSITION IX. THEOREM

41. *When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.*



GIVEN—PQ cutting QM and PN so that a and b on the same side of PQ are together equal to two right angles.

TO PROVE QM and PN parallel.

About O, the middle point of PQ, as a pivot, revolve the figure $QMXNP$ half way round to the symmetrical position $PM'X'N'Q$, so that P and Q exchange places.

The angle a is the supplement of b. Hyp.

Hence, when a takes the position a', PM' must be the prolongation of PN. § 29

[If two adjacent angles equal two right angles, their exterior sides form the same straight line.]

Likewise QN' is the prolongation of QM.

Now if these lines should meet on the right of PQ, as at X, they would also meet on the left, at X'. § 40

And we would have two straight lines between the two points, X and X', which is absurd. Ax. *a*

If they do *not* meet on the right of PQ, neither can they meet on the left of it. § 40

Hence QM and PN do not meet, and, being in the same plane, are parallel. Q. E. D.

It may be observed that the preceding proposition rests on only *two* of the three geometric axioms stated in § 10, viz.: the *superposition axiom*, assumed in turning the figure unchanged about O, and the *straight-line axiom*, used to prove that there cannot be two straight lines between X and X'. The *parallel axiom* (viz.: that through a point only one straight line can be drawn parallel to a given straight line) has only been used so far in Propositions VII. and VIII. Mathematicians have tried to dispense with the parallel axiom entirely, but have not succeeded. In fact, Lobatchewsky in 1829 proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from what we know it to be through experience. Lobatchewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, whereas the space in which we really live is called *Euclidean*, because Euclid (about 300 B.C.) first wrote a systematic geometry of our space. In Lobatchewsky's space, Proposition IX. would be true, but Propositions VII. and VIII. would not be true, nor would §§ 47, 48, 49, 51, 58, etc., in Book I., and §§ 284, 327, 329, etc., in Book III.

42. CONSTRUCTION. *To bisect a given straight line, AB.*

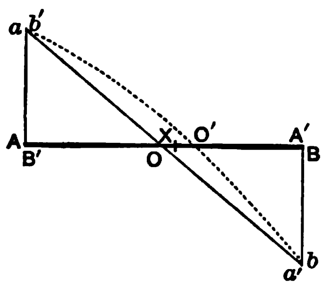


FIG. 1

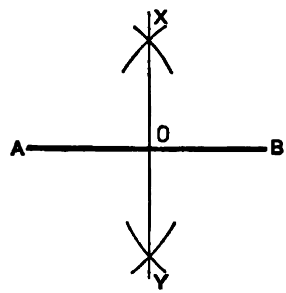


FIG. 2

First method (Fig. 1).—At A and B erect Aa and Bb equal perpendiculars on opposite sides of AB. Join ab cutting AB at O. O is the required middle point.

Proof.—Suppose the middle point of AB is not O, but some other point as X.

Then turn the whole figure about X until AX coincides with its equal BX, A falling on B (call this position of A, A'), and B on A (call this position of B, B'). And O will assume the position O' on the opposite side of X.

Then the perpendicular Aa will fall along Bb. § 18

[From a point in a straight line only one perpendicular can be drawn.]

And a will fall on b (call this position of a, a').

[Since Aa is equal to Bb.]

Likewise b will fall on a (call this position of b, b').

Then the straight line aOb takes the position $a'O'b'$.

That is, through two points, a and b, there would be two straight lines, which is absurd. Ax. *a*

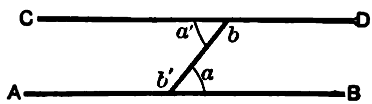
Hence the supposition that O is not the middle point is false, and O is the middle point. Q. E. D.

Second method (Fig. 2).—From A and B as centres with the same radius describe arcs intersecting at X and Y. Join XY intersecting AB at O, the required middle point.

[This method can be proved correct after reaching § 104.]

PROPOSITION X. THEOREM

43. *If two straight lines are cut by a third straight line, making the alternate interior angles equal, the lines are parallel.*



GIVEN $a = a'$.

TO PROVE AB and CD parallel.

$a' + b = 2$ right angles. § 22

[Being sup.-adj.]

Substitute for a' its equal a.

Then $a + b = 2$ right angles.

Therefore AB is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.] Q. E. D.

44. COR. I. *If two or more straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.*

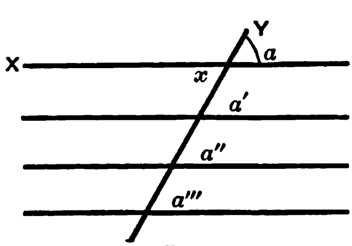


FIG. 1

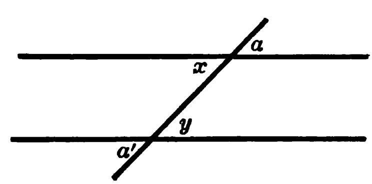


FIG. 2

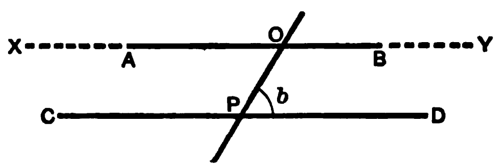
Hint.—Reduce to Proposition X. by means of Proposition V.

45. COR. II. *If two straight lines are cut by a third straight line so that the alternate exterior angles are equal, the lines are parallel.*

PROPOSITION XI. THEOREM

46. *If two parallel lines are cut by a third straight line, the sum of the two interior angles on the same side of the cutting line is two right angles.*

[Converse of Proposition IX.]



GIVEN—AB and CD parallel and cut by the straight line OP.
TO PROVE $b + POB = 2$ right angles.

Suppose XY to be a line drawn through O, making
$b + POY = 2$ right angles.

Then XY is parallel to CD. § 41

[When two straight lines are cut by a third straight line, if the two interior angles on the same side of the cutting line are together equal to two right angles, the two straight lines are parallel.]

But AB is parallel to CD. Hyp.
Hence AB coincides with XY. Ax. *b*

[Through a given point only one straight line can be drawn parallel to a given straight line.]

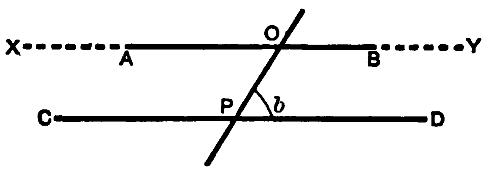
And $POB = POY$. Coinciding
Hence $b + POB = b \times POY$. Ax. 2
But $b + POY = 2$ right angles. Cons.
Hence $b + POB = 2$ right angles. Ax. 1

Q. E. D.

PROPOSITION XII. THEOREM

47. *If two parallel lines are cut by a third straight line, the alternate interior angles are equal.*

[Converse of Proposition X.]



GIVEN AB and CD parallel.

TO PROVE $b = AOP$.

Let XY be a line drawn through O, making $XOP = b$.

Then	XY is parallel to CD.	§ 43

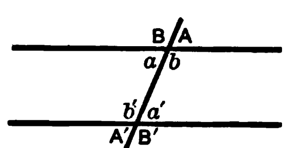
[If two straight lines are cut by a third straight line, making the alternate-interior angles equal, the lines are parallel.]

But	AB is parallel to CD.	Hyp.
Hence	AB coincides with XY.	Ax. *b*
And	$AOP = XOP$.	Coinciding
But	$b = XOP$.	Hyp.
Therefore	$AOP = b$.	Ax. I

Q. E. D.

48. COR. *If two or more parallel lines are cut by a third straight line, the corresponding angles are equal.*

49. *Remark.*—It follows from the previous propositions and corollaries that if two parallel lines are cut by a third straight line,



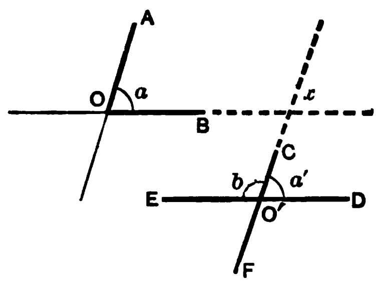
then $A = a = a' = A'$,

$$B = b = b' = B',$$

and any angle of the first set is supplementary to any angle of the second set.

PROPOSITION XIII. THEOREM

50. *Two angles whose sides are parallel, each to each, are either equal or supplementary.*



GIVEN—the angles at O and O' with their sides OA and OB respectively parallel to CF and ED.

TO PROVE the angle $a = a'$, and $a + b = 2$ right angles.

Produce OB and $O'C$ until they intersect.

Then $\left.\begin{matrix} a = x \\ a' = x \end{matrix}\right\}$ § 48

[Being corresponding angles of parallel lines.]

Therefore $a = a'$. Ax. 1

Moreover, $a' + b = 2$ right angles. § 22

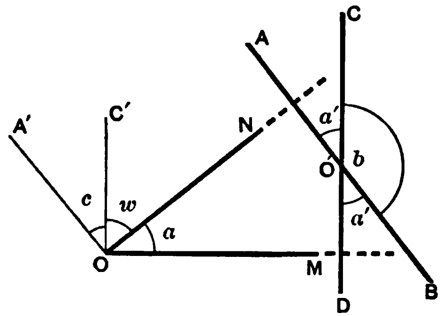
Substituting a for its equal a',

$a + b = 2$ right angles. Q. E. D.

51. *Remark.*—An angle viewed from its vertex has a right and a left side. If the sides of two angles are parallel, right to right and left to left, the angles are equal, if right to left the angles are supplementary.

PROPOSITION XIV. THEOREM

52. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



GIVEN—the angle NOM, or a, and the lines AB and CD intersecting at O' and respectively perpendicular to ON and OM.

TO PROVE—the angle $a=a'$, and $a+b=2$ right angles.

At O, draw OA' parallel to AB and OC' parallel to CD.

OA', being parallel to AB, is perpendicular to ON. § 36

[If two straight lines are parallel, and a third straight line is perpendicular to one of them, it is perpendicular to the other.]

For the same reason OC', being parallel to CD, is perpendicular to OM.

From each of the right angles $A'ON$ and $C'OM$ take away the common angle w.

This leaves $c=a$. Ax. 3

But $c=a'$. § 50

[Having their sides parallel, right to right and left to left.]

Therefore $a=a'$. Ax. 1

Moreover $a'+b=2$ right angles. § 22

[Being supplementary-adjacent.]

Substituting a for its equal a',

$a+b=2$ right angles. Q. E. D.

53. *Remark.*—If the sides of two angles are perpendicular, right to right and left to left, the angles are equal, if right to left the angles are supplementary.

TRIANGLES

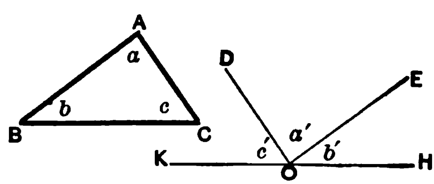
54. *Def.*—A **triangle** is a figure bounded by three straight lines called its **sides.**

55. *Def.*—A **right** triangle is a triangle one of whose angles is a right angle.

56. *Def.*—An **equiangular** triangle is one whose angles are all equal.

PROPOSITION XV. THEOREM

57. *The sum of the three angles of any triangle is two right angles.**



GIVEN ABC, any triangle, with a, b, and c its angles.

TO PROVE $a+b+c=2$ right angles.

Draw KH parallel to BC, and from O, any point of this line, draw OE and OD parallel respectively to the sides AB and AC.

Then
$$\left.\begin{aligned}a&=a'\\ b&=b'\\ c&=c'\end{aligned}\right\} \qquad \S\ 50$$

[Having their sides parallel, right to right, and left to left.]

* This was first proved by Pythagoras or his followers about 550 B.C.

Hence $a+b+c=a'+b'+c'$. Ax. 2

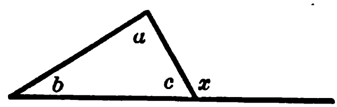
But $a'+b'+c'=2$ right angles. § 27

[The sum of all the angles about a point on one side of a straight line equals two right angles.]

Hence $a+b+c=2$ two right angles. Ax. I

Q. E. D.

58. COR. I. *If one side of a triangle be produced, the exterior angle thus formed equals the sum of the two opposite interior angles (and hence is greater than either of them).*



OUTLINE PROOF: $a+b+c=2$ right angles $=x+c$, whence $a+b=x$.
[Give reasons.]

59. COR. II. *If the sum of two angles of a triangle be given, the third angle may be found by taking the sum from two right angles.*

[What axiom applies?]

60. COR. III. *If two angles of one triangle are equal respectively to two angles of another triangle, the third angles will be equal.*

[What two axioms apply?]

61. COR. IV. *A triangle can have but one right angle or one obtuse angle.*

62. COR. V. *In a right triangle the sum of the two angles besides the right angle is equal to one right angle.*

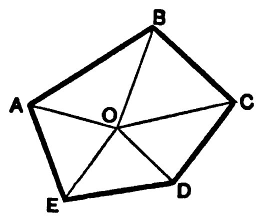
63. COR. VI. *In an equiangular triangle, each angle is one-third of two right angles, and hence two-thirds of one right angle.*

64. *Defs.*—A **polygon** is a figure bounded by straight lines called its **sides**.

A polygon is **convex** if no straight line can meet its sides in more than two points.

PROPOSITION XVI. THEOREM

65. *The sum of all the angles of any polygon is twice as many right angles as the figure has sides, less four right angles.*



GIVEN *ABCDE*, any polygon, having n sides.

TO PROVE—the sum of its angles is $2n-4$ right angles.

From any point O within the polygon draw lines to all the vertices forming n triangles.

The sum of the angles of each triangle is equal to 2 right angles. § 57

Hence the sum of the angles of the n triangles is equal to $2n$ right angles.

But the angles of the polygon make up all the angles of all the triangles except the angles about O, which make 4 right angles. § 28

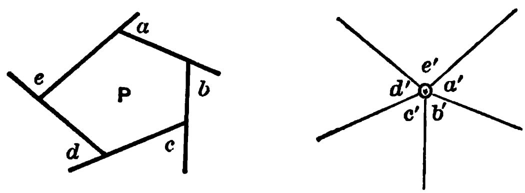
Hence the sum of the angles of the polygon is $2n-4$ right angles. Q. E. D.

66. *Defs.*—A **quadrilateral** is a polygon of four sides, a **pentagon,** of five, a **hexagon,** of six, an **octagon,** of eight, a **decagon,** of ten, a **dodecagon,** of twelve, a **pentedecagon,** of fifteen.

67. *Exercise.*— The sum of the angles of a quadrilateral equals what? of a pentagon? of a hexagon?

PROPOSITION XVII. THEOREM

68. *If the sides of any polygon be successively produced, forming one exterior angle at each vertex, the sum of these exterior angles is four right angles.*



GIVEN—the polygon P with successive exterior angles a, b, c, d, e.

TO PROVE $a+b+c+d+e=4$ right angles.

Through any point O draw lines successively parallel to the sides produced.

Then
$$\left.\begin{array}{r} a=a' \\ b=b' \\ c=c' \\ \text{etc.} \end{array}\right\} \qquad \S\ 50$$

[Two angles are equal if their sides are parallel and in the same order.]

Hence $a+b+c+\text{etc.}=a'+b'+c'+\text{etc.}$ Ax. 2

But $a'+b'+c'+\text{etc.}=4$ right angles. § 28

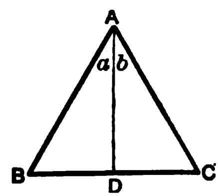
Therefore $a+b+c+\text{etc.}=4$ right angles. Ax. 1

Q. E. D.

69. *Defs.*— An **isosceles** triangle is a triangle two of whose sides are equal. The third side is called the **base.** The opposite vertex is called the **vertex** of the isosceles triangle, and the angle at that vertex the **vertex angle.** An **equilateral** triangle is one whose **three** sides are equal.

PROPOSITION XVIII. THEOREM

70. *The angles at the base of an isosceles triangle are equal.*



GIVEN—the isosceles triangle ABC, AB and AC being equal sides.

TO PROVE the angle B equals the angle C.

Suppose AD to be a line bisecting the angle A.

On AD as an axis revolve the figure ADC till it falls upon the plane of ADB.

AC will fall along AB.

[Since angle $a = b$, by construction.]

C will fall on B.

[Since $AB = AC$, by hypothesis.]

DB will coincide with DC. Ax. a

[Their extremities being the same points.]

Hence angle B = angle C. § 15

[Since they coincide.] Q. E. D.

71. COR. I. *The line which bisects the vertex angle of an isosceles triangle bisects the base.*

Hint.—Show where this was proved in the preceding demonstration.

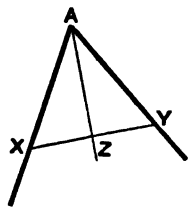
72. COR. II. *The line joining the middle point of the base with the vertex of an isosceles triangle bisects the vertex angle.*

Hint.—Prove that the line which bisects the vertex angle coincides with the given line.

73. COR. III. *Every equilateral triangle is also equiangular, and each angle is one-third of two right angles.*

Question.—In how many different ways is an equilateral triangle isosceles?

74. CONSTRUCTION. *To bisect any given angle A.*

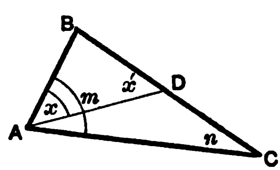


On the sides of the angle, lay off $AX = AY$. Join X and Y. Bisect XY at Z (§ 42). Join A and Z. AZ will bisect the angle A.

Hint.—To prove this method correct apply § 72.

PROPOSITION XIX. THEOREM

75. *If two sides of a triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.*



GIVEN in the triangle ABC the side $BC >$ side AB.

TO PROVE the angle $m >$ angle n

On BC take $BD = BA$, and join A and D.

Then $x = x'$. § 70

[Being base angles of an isosceles triangle.]

But $x' > n$. § 58

[An exterior angle of a triangle (ADC) is greater than either of the opposite interior angles.]

Substituting x for x'. $x > n$.

But $m>x$. Ax. 10

Hence $m>n$ Ax. 13

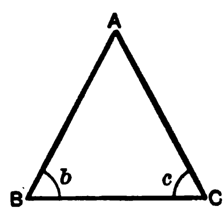
Q. E. D.

OUTLINE PROOF: $m>x=x'>n$, hence $m>n$.

PROPOSITION XX. THEOREM

76. *If two angles of a triangle are equal, the sides opposite are equal—that is, the triangle is isosceles.*

[Converse of Proposition XVIII.]



GIVEN in the triangle ABC, the angle $b=c$.

TO PROVE side AC=side AB.

If AC and AB were unequal, b and c would be unequal. § 75

[If two sides of a triangle are unequal the opposite angles are unequal, etc.]

But this contradicts the hypothesis that angle b=angle c.

Hence $AC=AB$ Q. E. D.

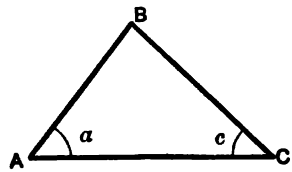
PROPOSITION XXI. THEOREM

77. *If two angles of a triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.*

[Converse of Proposition XIX.]

GIVEN in the triangle ABC, the angle $a>$angle c.

TO PROVE side $BC>$side AB.



Either AB is equal to, greater than, or less than BC.

If $AB=BC$, then would $c=a$. § 70

[The angles at the base of an isosceles triangle are equal.]

If $AB>BC$, then would $c>a$. § 75

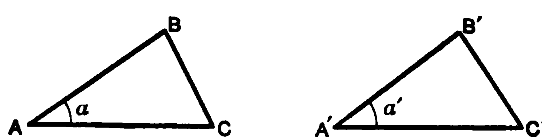
[If two sides of a triangle are unequal, the opposite angles are unequal, etc.]

But both of these conclusions contradict the hypothesis that angle $a>$ angle c.

Therefore $AB>BC$ Q. E. D.

PROPOSITION XXII. THEOREM

78. *If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal.*



GIVEN — AB, AC, and a, of the triangle ABC respectively equal to $A'B'$, $A'C'$, and a', of the triangle $A'B'C'$.

TO PROVE the two triangles are equal.

Place ABC on $A'B'C'$, making AB coincide with its equal $A'B'$.

Then, since $a=a'$, the side AC will fall along $A'C'$.

Also, since $AC=A'C'$, the point C will fall on C'.

Then BC will coincide with $B'C'$. Ax. *a*

[Having their extremities in the same points.]

Hence the triangles completely coincide and are equal. § 15

Q. E. D.

79. CONSTRUCTION. *To construct an angle at a given point A' as its vertex, and on a given line $A'B'$ as a side, equal to a given angle BAC at a different vertex A.*

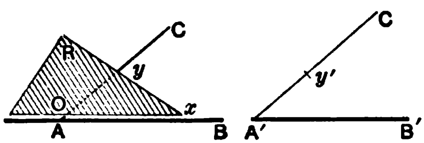


FIG. 1

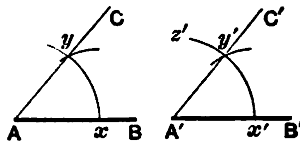


FIG. 2

First method (Fig. 1). — Place a triangular ruler, R, so that the straight edge falls along AB. Mark y on another edge where this edge cuts AC. Also mark the point A on the ruler and call it O. Draw Oy on the ruler. Then the angle BAC is reproduced on the ruler as xOy. Then, placing the ruler with O at A' and Ox along $A'B'$, retransfer the angle xOy of the ruler to the paper making $B'A'C'$. Then $B'A'C' = BAC$.

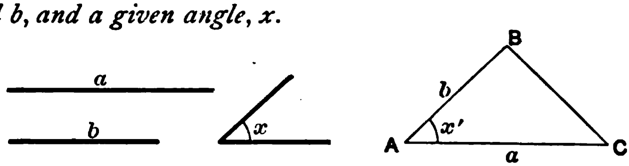
Which geometric axiom and which general axiom apply?

Evidently any piece of paper with a straight edge will serve as the ruler.

Second method (Fig. 2).—With A as a centre and any convenient radius describe an arc xy. With A' as a centre and the same radius describe the indefinite arc $x'z'$. Then take xy as a radius, and with x' as a centre describe an arc intersecting $x'z'$ at y'. Join y' and A'. $y'A'B'$ is the angle required.

This cannot be proved until reaching § 88.

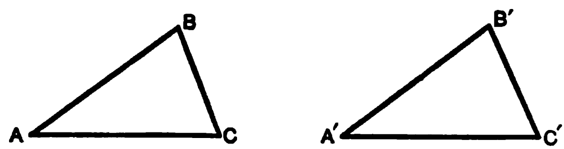
80. CONSTRUCTION. *To construct a triangle with two sides and the included angle equal respectively to two lines, a and b, and a given angle, x.*



Lay off $AC=a$. Make $x'=x$ (§ 79). Lay off $AB=b$. Join BC. ABC is the triangle required.

PROPOSITION XXIII. THEOREM

81. *If two triangles have a side and two adjacent angles of one equal to a side and two adjacent angles of the other, the two triangles are equal.*



GIVEN—in the two triangles ABC and $A'B'C'$, $AB=A'B'$, and the angles A and B equal respectively A' and B'.

TO PROVE the triangles are equal.

Apply ABC to $A'B'C'$, making AB coincide with $A'B'$.

Then AC will fall along $A'C'$, and likewise BC along $B'C'$.

[Since the angles A and B respectively equal A' and B'.]

Hence C must fall somewhere on $A'C'$, and likewise somewhere on $B'C'$.

It must therefore fall on their intersection C'.

And, since the triangles completely coincide, they are equal. Q. E. D.

82. COR. I. *If two triangles have a side and any two angles of one equal respectively to a side and two similarly situated angles of the other, the triangles are equal.*

Hint.—Reduce to the preceding Proposition by § 59.

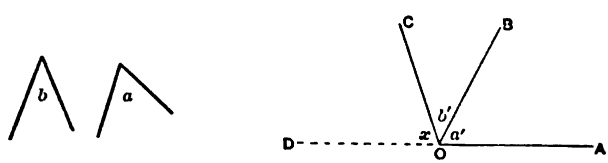
Question.—In how many ways can ABC and $A'B'C'$ have a side and two similarly situated angles equal? Draw two triangles having a side and two angles of each equal but without having the angles similarly situated.

83. *Defs.*—The **hypotenuse** of a right triangle is the side opposite the right angle. The other sides are the **perpendicular sides.**

84. COR. II. *Two right triangles are equal, if the hypotenuse and an acute angle of one are respectively equal to the hypotenuse and an acute angle of the other.*

85. COR. III. *Two right triangles are equal, if a perpendicular side and an acute angle of one are respectively equal to a perpendicular side and the similarly situated acute angle of the other.*

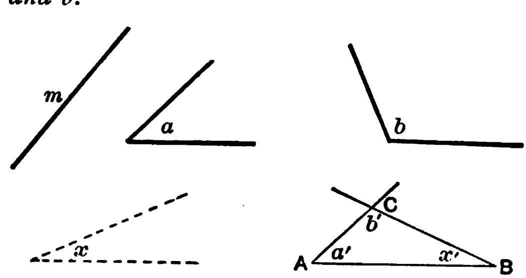
86. CONSTRUCTION. *If two angles of a triangle are equal to given angles a and b, to find the third angle.*



On any line OA construct angle $a'=a$, and on OB at the same vertex O construct $b'=b$. Produce OA to D making the angle x with OC. x is the angle required.

[Prove by § 59.]

87. CONSTRUCTION. *To construct a triangle with a side and two angles equal respectively to a given line m and two angles a and b.*



Find (by § 86) x the third angle of the triangle.

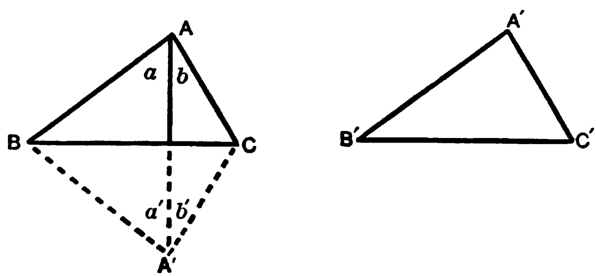
Draw any straight line AB equal to m, and at A and B construct whichever two angles of the three, a, b, x, be required to be adjacent to the given side. If the constructed sides of these angles produced meet, let C be the point of intersection. ABC is the triangle required. For AB equals m by construction, and the angles a' and b' equal a and b by construction or by proof (§ 59).

Discussion.—This problem is impossible if the two given angles are together equal to or greater than two right angles (by § 57).

Question.—Is the problem of § 80 ever impossible?

PROPOSITION XXIV. THEOREM

88. *If two triangles have their three sides respectively equal, they are equal.*



GIVEN—in the triangles ABC and $A'B'C'$, $AB=A'B'$, $BC=B'C'$, and $AC=A'C'$.

TO PROVE triangle $ABC=$ triangle $A'B'C'$.

Place $A'B'C'$ so that $B'C'$ shall coincide with its equal BC, but A' shall fall on the side of BC opposite A, and join AA'.

The triangle ABA' has $AB=A'B$, that is, is isosceles. Hyp.

Hence $a=a'$. § 70

[Being base angles of an isosceles triangle.]

Likewise we may prove $b=b'$.

Adding $a+b=a'+b'$. Ax. 2

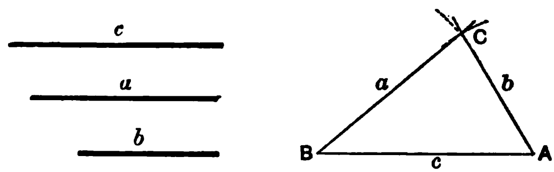
Or angle A = angle A'.

Hence triangle ABC = triangle $A'B'C'$. § 78

[Having two sides and the included angles equal.]

Q. E. D.

89. CONSTRUCTION. *To construct a triangle with its three sides equal to given lines a, b, and c.*



Draw AB equal to c. From A as a centre and with b as a radius describe an arc. From B as a centre with a as a radius describe another arc. If these arcs intersect join C, their intersection, with A and B. ABC is the required triangle.

Discussion.—The problem is impossible if one of the given lines is equal to or greater than the sum of the other two.

90. *Exercise.*—By Proposition XXIV. prove that each of the following constructions is correct:

(1.) For erecting a perpendicular, as in § 21, second method.

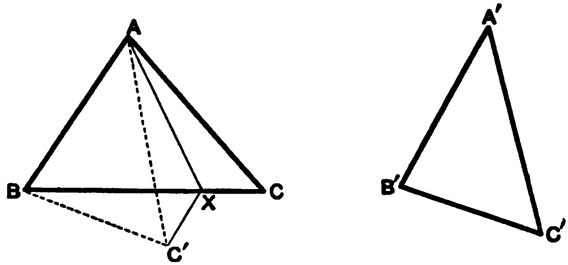
(2.) For making an angle equal to a given angle, as in § 79 second method.

Question.—If two quadrilaterals have their sides equal, each to each, are they necessarily equal?

Question.—In stating Proposition XXIV. does it matter in what order the sides are arranged?

PROPOSITION XXV. THEOREM

91. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



GIVEN—two triangles, ABC and $A'B'C'$, having $AB = A'B'$ and $AC = A'C'$, but angle $A >$ angle A'.

TO PROVE $BC > B'C'$

Apply $A'B'C'$ and ABC making $A'B'$ coincide with its equal AB.

The angle A' will fall within the angle BAC.

Draw AX bisecting the angle CAC' and meeting BC in X. Join C' and X.

In the two triangles ACX and $AC'X$

$AC = AC'$, Hyp.

$AX = AX$, Iden.

angle $CAX =$ angle $C'AX$. Cons.

Hence triangle $ACX =$ triangle $AC'X$. § 78

[If two triangles have two sides, and the included angle of the one equal respectively to two sides and the included angle of the other, the triangles are equal.]

Hence $XC = XC'$.

Now $BC' < BX + XC'$. § 7

[A straight line is the shortest path between any two of its points.]

Substituting XC for its equal XC',

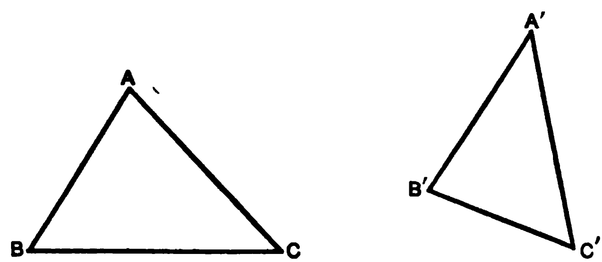
$BC' < BX + XC$.

Or $BC' < BC$. Q. E. D.

PROPOSITION XXVI. THEOREM

92. *If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*

[Converse of Proposition XXV.]



GIVEN—two triangles ABC and $A'B'C'$, having $AB=A'B'$ and $AC=A'C'$, but $BC>B'C'$.

TO PROVE angle $A>$ angle A'.

Angle A is either equal to, less than, or greater than angle A'.

If $A=A'$, then would $BC=B'C'$. § 78

[Triangles having two sides and the included angle respectively equal are equal.]

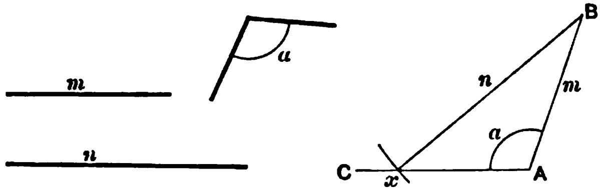
If $A<A'$ then would $BC<B'C'$. § 91

[If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.]

But both these conclusions contradict the hypothesis.

Therefore $A>A'$. Q. E. D.

93. CONSTRUCTION. *To construct a triangle when two sides, m and n, and an angle opposite one of them, a, are given.*

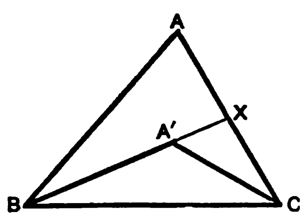


By § 79 construct the given angle a at any vertex A. On one of its sides lay off AB equal to m. From B as a centre with n as a radius draw an arc intersecting the other side at x. ABx is the triangle required.

Question. Is the problem ever impossible? Are there ever *two* triangles which satisfy the conditions?

PROPOSITION XXVII. THEOREM

94. *If from a point within a triangle two straight lines are drawn to the extremities of one side, their sum will be less than the sum of the other two sides of the triangle.*



GIVEN—the triangle ABC and the lines $A'B$ and $A'C$ drawn from A' to the extremities of BC.

TO PROVE $A'B + A'C < AB + AC$.

Prolong BA' to meet AC at X.

Then $A'C < A'X + XC$. § 7

And also $A'B + A'X < XA + AB$. § 7

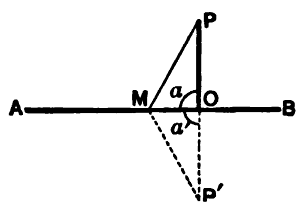
Adding, $A'C + A'B + \underline{A'X} < \underline{A'X} + \overline{XC + XA} + AB$. Ax. 9

Cancel $A'X$ from each side and substitute AC for $XC + XA$.

Then $A'C + A'B < AC + AB$. Q. E. D.

PROPOSITION XXVIII. THEOREM

95. *The perpendicular is the shortest line between a point and a straight line.*



GIVEN—PO the perpendicular from a point P to a straight line AB and PM *any* oblique line from P to AB.

TO PROVE $PO<PM$.

Revolve PMO about AB to form the symmetrical figure $P'MO$. § 32

Then $PO=P'O$ and $PM=P'M$.

Also PO and $P'O$ form a straight line § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PP'<PM+MP'$. § 7

Or $2\ PO<2\ PM$.

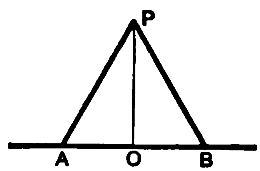
Whence $PO<PM$. Ax. 8

Q. E. D.

96. *Def.*—The "distance" from a point to a straight line means the **shortest** distance, and hence the **perpendicular** distance.

PROPOSITION XXIX. THEOREM

97. *Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.*



GIVEN—PO perpendicular to AB, and PA and PB drawn from P cutting off $AO=BO$.

TO PROVE $PA=PB$.

In the *right* triangles POA and POB

$PO=PO$. Iden.

$AO=BO$. Hyp.

Hence triangle POA = triangle POB. § 78

[Having two sides and included angle respectively equal.]

Therefore $PA=PB$

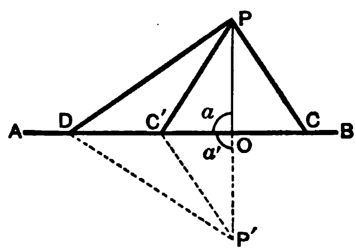
[Being homologous sides of equal triangles.] Q. E. D.

98. *Exercise.*—The sum of the three lines from any point within a triangle to the three vertices is less than the sum of the three sides, but greater than half their sum.

Hint.—Apply §§ 7 and 94.

PROPOSITION XXX. THEOREM

99. ***Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot, the more remote is the greater.***



GIVEN PO perpendicular to AB, and OC less than OD.

TO PROVE $PC < PD$.

Take $OC' = OC$ and join PC'.

Then $PC' = PC$. § 97

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

Revolve the figure about AB forming the symmetrical figure $P'DO$.

Then PO and OP' form the same straight line. § 29

[If two adjacent angles (a and a') are together two right angles, their exterior sides form a straight line.]

Now $PC' + P'C' < PD + P'D$. § 94

[If from a point within a triangle straight lines are drawn to the extremities of one side, the sum will be less than the sum of the other two sides.]

Substitute PC' for its equal impression $P'C'$, and likewise PD for $P'D$.

Then $2\,PC' < 2\,PD$.

Whence $PC' < PD$. Ax. 8

Substituting PC for PC', $PC < PD$. Q. E. D.

PROPOSITION XXXI. THEOREM

100. *If from a point in a perpendicular to a given straight line two equal oblique lines are drawn, they cut off equal distances from the foot of the perpendicular, and of two unequal oblique lines the greater cuts off the greater distance.*

[Converse of Proposition XXX.]

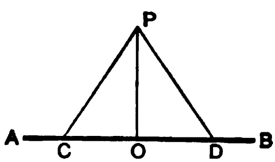


FIG. 1

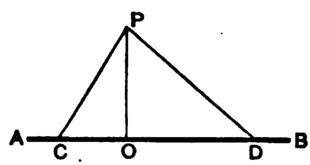


FIG. 2

I. GIVEN PO perpendicular to AB, and $PC = PD$. [Fig. 1.]

TO PROVE $OC = OD$.

OC is either greater than, less than, or equal to OD.

If $OC > OD$, then would $PC > PD$.
If $OC < OD$, then would $PC < PD$. § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OC = OD$. Q. E. D.

II. GIVEN PO perpendicular to AB and $PD > PC$. [Fig. 2.]

TO PROVE $OD > OC$.

OD is either equal to, less than, or greater than OC.

If $OD = OC$, then would $PD = PC$. § 97

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

If $OD<OC$, then would $PD<PC$. § 99

[Of two oblique lines drawn from the same point in a perpendicular and cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.]

But both these conclusions contradict the hypothesis.

Therefore $OD>OC$. Q. E. D.

101. COR. *Two right triangles are equal if they have the hypotenuse and a side of one equal to the hypotenuse and a side of the other.*

102. *Def.*—A line is the **locus** of all points which satisfy a given condition, if all points in that line satisfy the condition, and no points out of that line satisfy it.

Question.—What is the locus of all points three inches from a given point? Prove it.

PROPOSITION XXXII. THEOREM

103. *The locus of all points equally distant from two given points is a straight line bisecting at right angles the line joining the given points.*

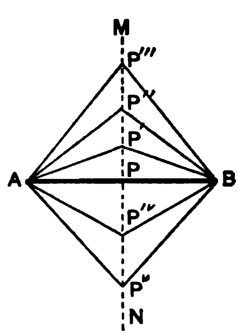


FIG. 1

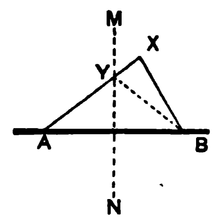


FIG. 2

GIVEN A and B, two fixed points.

TO PROVE—that the locus of all points equally distant from A and B is a straight line MN, perpendicular to AB at its middle point, P.

It is necessary to prove:

I. Every point in MN satisfies the condition of being equally distant from A and B.

II. No point without MN satisfies this condition.

I. (Fig. 1.) Draw MN perpendicular to AB at its middle point, and let P', P'', P''', P^{IV}, etc., be any points in MN.

Then $AP=PB$. Cons.

Hence $P'A=P'B$; $P''A=P''B$; $P'''A=P'''B$, etc. § 97

[Two oblique lines drawn from the same point in a perpendicular cutting off equal distances from the foot of the perpendicular are equal.]

That is, every point in MN is equally distant from A and B.

II. (Fig. 2.) Let X be any point without MN.

Draw XA and XB. One of these lines must cut MN in some point as Y.

Then $XB<XY+YB$. § 7

But $YA=YB$. § 97

Substituting YA for YB, $XB<XY+YA$.

Or $XB<XA$.

Hence every point without MN is unequally distant from A and B. Q. E. D.

104. COR. *Two points equally distant from the extremities of a straight line determine a perpendicular bisector to that line.*

105. *Exercise.*—Show that the following methods of construction were correct:

(1.) Of dropping a perpendicular, as in § 35, second method.

(2.) Of bisecting a straight line, as in § 42, second method.

106. *Exercise.*—The bisector of an angle is the locus of all points within the angle equally distant from its sides.

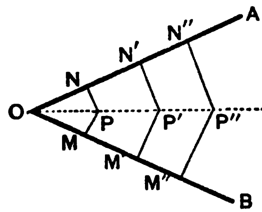


FIG. 1

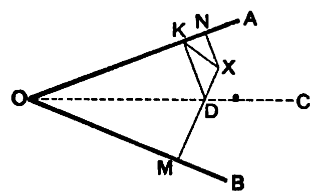


FIG. 2

Hint.—It is necessary to prove that every point in the bisector OC is equidistant from the sides of the angle, and that any point X (Fig. 2) without the bisector is unequally distant from the sides of the angle.

107. *Exercise.*—The three perpendicular bisectors of the sides of a triangle meet in a common point.

Hint.—The intersection, O (Fig. 1), of MM' and NN' is equidistant from A and B, and from A and C. § 103

Hence O is equidistant from B and C.

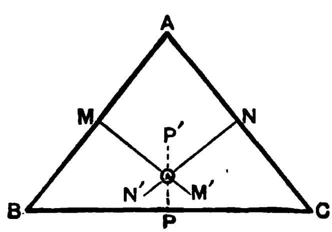


FIG. 1

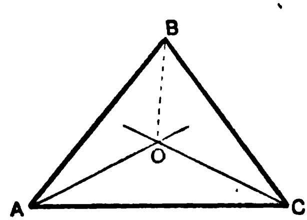


FIG. 2

108. *Exercise.*—The three bisectors of the angles of a triangle meet in a common point.

Hint.—The intersection, O (Fig. 2), of AO and CO is equidistant from AB and AC, and from AC and CB. § 106

Hence O is equidistant from AB and CB.

109. *Exercise.*—What is the locus of all points equally distant from two intersecting straight lines?

110. *Exercise.*—What is the locus of all points at a given distance from a fixed straight line of indefinite length?

111. *Exercise.*—What is the locus of all points at a given distance from a given line of a definite length?

PARALLELOGRAMS

112. *Defs.*—A **parallelogram** is a quadrilateral whose opposite sides are parallel.

A **rhombus** is a parallelogram whose sides are all equal.

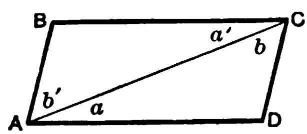
A **rectangle** is a parallelogram whose angles are all right angles.

A **square** is a rectangle whose sides are all equal.

113. *Def.*—A **diagonal** of a quadrilateral is a straight line joining opposite vertices.

PROPOSITION XXXIII. THEOREM

114. ***A diagonal of a parallelogram divides it into two equal triangles.***



GIVEN the parallelogram $ABCD$ and the diagonal AC.

TO PROVE—that the triangles ABC and ACD are equal.

In the triangles ABC and ADC

$AC=AC$, Iden.

$\left.\begin{array}{l} a=a', \\ b=b'. \end{array}\right\}$ § 47

[Being alt.-int. angles of parallel lines.]

Hence triangle $ABC=$triangle ADC. § 81

[Having a side and two adjacent angles in each respectively equal.]

Q. E. D.

115. COR. I. *In any parallelogram the opposite sides are equal and the opposite angles are equal.*

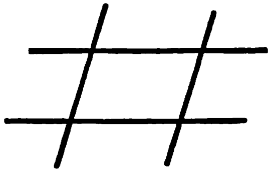


FIG. 1

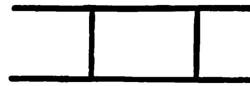


FIG. 2

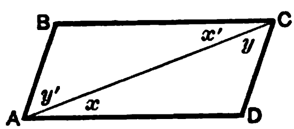
116. COR. II.—*Parallels comprehended between parallels are equal* [Fig. 1].

117. COR. III. *Parallels are everywhere equally distant* [Fig. 2].

Hint.—Apply §§ 33, 36, 116.

PROPOSITION XXXIV. THEOREM

118. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



GIVEN—any quadrilateral having its opposite sides equal, viz.: $AB = CD$ and $AD = BC$.

TO PROVE the quadrilateral is a parallelogram.

Draw the diagonal AC.

$AC = AC$. Iden.

$\left.\begin{array}{l} AB = CD. \\ AD = BC. \end{array}\right\}$ Hyp.

Hence triangle ABC = triangle ACD. § 88

[Having three sides respectively equal.]

And $x = x'$.

[Being homologous angles of equal triangles.]

Therefore BC is parallel to AD. § 43

[When two straight lines (BC and AD) are cut by a third straight line (AC) making the alternate interior angles (x and x') equal, the straight lines are parallel.]

In like manner, using y and y', we may prove AB parallel to CD.

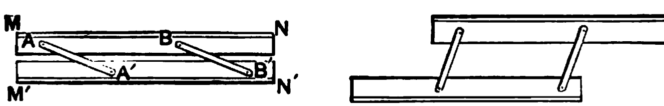
Therefore $ABCD$, having its opposite sides parallel, is a parallelogram. Q. E. D.

119. A "parallel ruler" is formed by two rulers (MN and $M'N'$), usually of wood pivoted to two metal strips (AA' and BB'), under the following conditions:

(1.) The distances on the rulers between pivots are equal; i. e., $AB = A'B'$.

(2.) The distances on the strips between pivots are equal; i. e., $AA' = BB'$.

(3.) In each ruler the edge is parallel to the line of pivots; i. e., AB is parallel to MN, and $A'B'$ is parallel to $M'N'$.



120. *Exercise.*—Prove: (1) the quadrilateral whose vertices are the pivots (i. e., the figure $ABB'A'$) is always a parallelogram, whether the ruler be closed or opened.

(2.) The edges of the rulers are always parallel (i. e., MN and $M'N'$ are parallel).

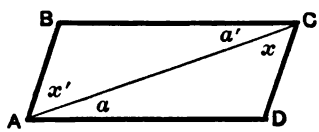
121. *Exercise.*—Show how to use the parallel ruler for drawing a straight line through a given point parallel to a given straight line, and prove the method correct.

Extend the method so as to apply even when the point is at a great distance from the line.

122. *Exercise.* — The diagonals of a parallelogram bisect each other.

PROPOSITION XXXV. THEOREM

123. *A quadrilateral which has two of its sides equal and parallel is a parallelogram.*



GIVEN — the quadrilateral $ABCD$ having BC equal and parallel to AD.

TO PROVE $ABCD$ is a parallelogram.

Draw the diagonal AC.

In the triangles ABC and ACD,

$AC = AC$, Iden.

$AD = BC$, Hyp.

angle a = angle a'. § 47

[Being alt. int. angles.]

Therefore triangle ABC = triangle ACD. § 78

[Having two sides and the included angle respectively equal.]

Hence $x = x'$.

[Being homologous angles of equal triangles.]

Hence AB is parallel to CD. § 43

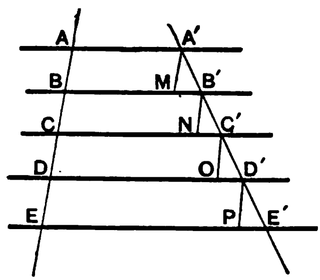
[When two straight lines are cut by a third straight line, making the alt. int. angles equal, the lines are parallel.]

Therefore $ABCD$ is a parallelogram.

[Having its opposite sides parallel.] Q. E. D.

PROPOSITION XXXVI. THEOREM

124. *If any number of parallels intercept equal parts on one cutting line, they intercept equal parts on every other cutting line.*



GIVEN—AA', BB', CC', DD', EE', any number of parallel lines cutting off the equal parts AB, BC, CD, DE, on AE.

TO PROVE—the parts on any other line $A'E'$ are also equal, viz.: $A'B'$, $B'C'$, $C'D'$, $D'E'$.

Construct parallels to AE through the points A', B', C', D'.

Then $AB=A'M;\ BC=B'N;$ etc. § 116

[Parallels comprehended between parallels are equal.]

But $AB=BC=$ etc. Hyp.

Therefore $A'M=B'N=$ etc. Ax. 1

Also, in the triangles $A'MB'$, $B'NC'$, etc.,

angle $A'=$ angle $B'=$ etc. § 48

[Being corresponding angles of parallels.]

And angle $M=$ angle $N=$ etc. § 50

[Having their sides parallel and in the same order.]

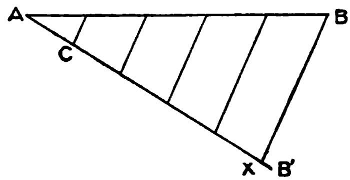
Hence triangle $A'MB'=$ triangle $B'NC'=$ etc. § 82

[Having a side and two angles respectively equal.]

Hence $A'B'=B'C'=C'D'=D'E'$.

[Being homologous sides of equal triangles.] Q. E. D.

125. CONSTRUCTION. *To divide a given line AB into any number of equal parts.*



From A draw any indefinite line AB' and lay off upon it any length AC.

Apply AC the required number of times on AB' and suppose X to be the last point of division. Join XB.

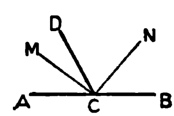
From the various points of division draw parallels to XB.

These parallels will cut AB in the required points of division.

Prove this method correct by Proposition XXXVI.

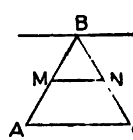
PROBLEMS

126. *Exercise.* — The bisectors of two supplementary-adjacent angles are perpendicular.



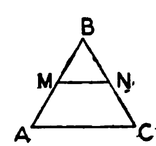
127. *Exercise.*—A straight line parallel to the base of a triangle and bisecting one side bisects the other also.

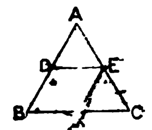
Hint.—Apply § 124.



128. *Exercise.*—A straight line joining the middle points of two sides of a triangle is parallel to the third side.

Hint.—Show that this line coincides with a line drawn as in § 127.



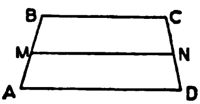


129. *Exercise.*—A straight line joining the middle points of two sides of a triangle is equal to half the third side.

Hint.—Prove $DE = BX$, and $DE = XC$.

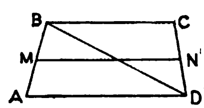
130. *Defs.*—A **trapezoid** is a quadrilateral, two of whose sides are parallel.

The parallel sides are called the **bases**.



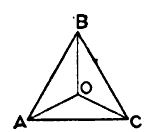
131. *Exercise.*—A straight line parallel to the bases of a trapezoid and bisecting one of the remaining sides bisects the other also.

132. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the bases.



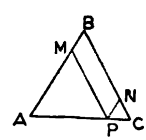
133. *Exercise.*—A straight line joining the middle points of the two non-parallel sides of a trapezoid is equal to half the sum of the bases.

134. *Exercise.*—Any side of a triangle is greater than the difference of the other two.

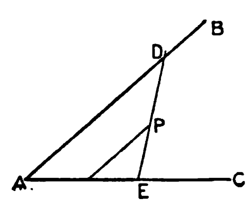


135. *Exercise.*—The sum of the three lines from any point within a triangle to the three vertices is less than the sum of the three sides, but greater than half their sum.

Hint.—Apply §§ 7 and 94.

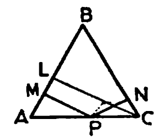


136. *Exercise.*—If from a point in the base of an isosceles triangle parallels to the sides are drawn, a parallelogram is formed, the sum of whose four sides is the same wherever the point is situated (and is equal to the sum of the equal sides).



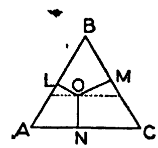
137. *Exercise.*—Given any angle A and any point P within it. Show a method of drawing a line through P to the sides of the angle which shall be bisected at P.

138. *Exercise.*—If from a point in the base of an isosceles triangle perpendiculars to the sides are drawn, their sum is the same wherever the point is situated (and is equal to the perpendicular from one extremity of the base to the opposite side).



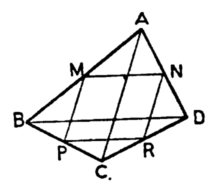
139. *Exercise.*—If from a point within an equilateral triangle perpendiculars to the three sides are drawn, the sum of these lines is the same wherever this point is situated (and is equal to the perpendicular from any vertex to the opposite side).

Hint.—Apply § 138.



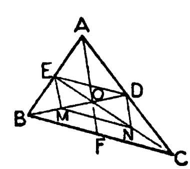
140. *Exercise.*—The straight lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.

Hint.—Apply § 128.



141. *Def.*—A **median** of a triangle is a straight line from a vertex to the middle point of the opposite side.

142. *Exercise.*—The three medians of any triangle intersect in a common point which is two-thirds of the distance from each vertex to the middle of the opposite side.



Hint.—*Two* of these lines, CE and BD, meet at some point O.

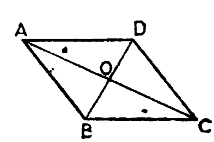
Take M and N, the middle points of BO and CO.

Draw $EDNM$. Prove it is a parallelogram by proving ED and MN each parallel to and equal to half of BC.

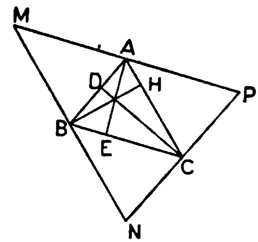
Then prove $OE = ON = NC$, and $DO = OM = MB$.

Thus we have proved that one of the medians, as BD, is cut by another, CE, at a point two-thirds of its length from B. We may likewise prove that it is also cut by the third median in the same point. Hence, etc.

143. *Exercise.*—The diagonals of a rhombus bisect each other at right angles, and also bisect the angles of the rhombus.



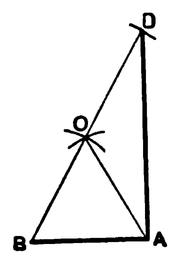
144. *Exercise.*—The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Hint.—Draw through each vertex a parallel to the opposite side. Prove AE, BH, and CD are perpendicular bisectors of the sides of the new triangle MNP, and apply § 107.

145. *Exercise.*—Prove that the following is a correct method for erecting a perpendicular from a point A in a line AB.

With A as a centre describe an arc. With the same radius and any other point, B, in the line as a centre, describe a second arc intersecting the first at O. With O as a centre and the same radius describe a third arc. Join B and O and produce BO to meet the third arc at D. Then AD is the perpendicular required.



Hint.—Of the four right angles of the two triangles, two are at O. Show that half the remainder are at A.

PLANE GEOMETRY

BOOK II

THE CIRCLE

146.* *Def.*—A **circle** is a plane figure bounded by a line all points of which are equally distant from a point within called the **centre.**

147. *Defs.*—The line which bounds the circle is called its **circumference.**

An **arc** is any part of a circumference.

148.* *Def.*—Any straight line from the centre to the circumference is a **radius.**

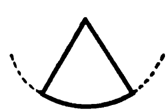
By the definition of a circle all its radii are equal.

149. *Def.*—A **chord** is a straight line having its extremities in the circumference.

150. *Def.*—A **diameter** is a chord through the centre.

All diameters are equal, each being twice a radius.

151. *Defs.*—A **sector** is that portion of a circle bounded by two radii and the intercepted arc.



SECTOR

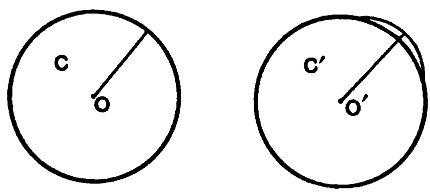
The angle between the radii is called the **angle** of the **sector.**

152. *Def.*—**Concentric** circles are circles which have the same centre.

* These definitions are repeated from § 20.

PROPOSITION I. THEOREM

153. *Circles which have equal radii are equal, and if their centres be made to coincide they will coincide throughout; conversely, equal circles have equal radii.*



I. GIVEN—any two circles, C and C' with centres O and O' and equal radii.

TO PROVE the circles C and C' are equal.

Place the circles so that O falls on O'.

Then the circumference of C will coincide with the circumference of C'.

For, if any portion of one fell without the other, its distance from the centre would be greater than the distance of the other. Hence the radii would be unequal, which is contrary to the hypothesis. Ax. 10

Therefore, the circumferences coincide, and the circles coincide and are equal. Q. E. D.

II. CONVERSELY:

GIVEN two equal circles.

TO PROVE their radii equal.

Since the circles are equal they can be made to coincide, and therefore their radii will coincide, and are equal. Q. E. D.

154. COR. I. Hence, *if a circle be turned about its centre as a pivot, its circumference will continue to occupy the same position.*

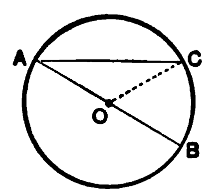
155. COR. II. *The diameter of a circle bisects the circle and the circumference.*

Hint.—Fold over on the diameter as an axis.

156. *Defs.*—The halves into which a diameter divides a circle are called **semicircles,** and the halves into which it divides the circumference are called **semicircumferences.**

PROPOSITION II. THEOREM

157. *The diameter of a circle is greater than any other chord.*



GIVEN—the circle ABC and AC, any chord not a diameter.

TO PROVE $AC <$ diameter AB.

Draw the radius OC.

$AC < AO + OC$. § 7

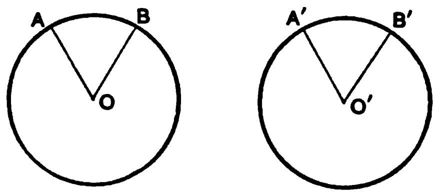
Substitute for OC the equal radius OB.

Then $AC < AO + OB$.

That is $AC < AB$. Q. E. D.

PROPOSITION III. THEOREM

158. *In the same circle or equal circles, equal angles at the centre intercept equal arcs; conversely, equal arcs are intercepted by equal angles at the centre.*



I. GIVEN—equal circles and equal angles at their centres, O and O'.

TO PROVE arc $AB = A'B'$.

Apply the circles making the angle O coincide with angle O'.

A will coincide with A', and B with B'. § 153

[For $AO = A'O'$, and $OB = O'B'$, being radii of equal circles.]

Then the arc AB will coincide with the arc $A'B'$, and is equal to it. § 153

Q. E. D.

II. CONVERSELY:

GIVEN—equal circles having equal arcs AB and $A'B'$.

TO PROVE—the subtended angles O and O' equal.

Apply the circles making the arc AB coincide with its equal $A'B'$.

Then AO will coincide with $A'O'$, and BO with $B'O'$. Ax. *a*

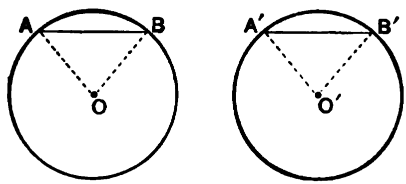
Therefore angles O and O' coincide and are equal. Q. E. D.

159. *Exercise.*—In the same circle or equal circles equal angles at the centre include equal sectors, and conversely.

The proof is analogous to § 158, requiring "sector" in place of "arc."

PROPOSITION IV. THEOREM

160. *In the same circle or equal circles, equal chords subtend equal arcs; conversely, equal arcs are subtended by equal chords.*



GIVEN—equal circles, O and O', and equal chords, AB and $A'B'$.

TO PROVE arc AB = arc $A'B'$.

Draw the four radii OA, OB, $O'A'$, $O'B'$.

In the triangles AOB and $A'O'B'$

$AB = A'B'$. Hyp.

$AO = A'O'$, and $OB = O'B'$. § 153

[Being radii of equal circles.]

Hence triangle AOB = triangle $A'O'B'$. § 88

[Having three sides respectively equal.]

Hence angle O = angle O'.

[Being corresponding angles of equal triangles.]

Therefore arc AB = arc $A'B'$. § 158

Q. E. D.

CONVERSELY:

GIVEN—equal circles O and O' and arc AB = arc $A'B'$.

TO PROVE chord AB = chord $A'B'$.

Since the arcs are equal, angle O = angle O'. § 158

And the four radii are equal. § 153

Hence triangle AOB = triangle $A'O'B'$. § 78

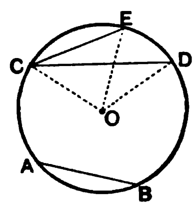
[Having two sides and the included angle respectively equal.]

Therefore chord AB = chord $A'B'$.

[Being corresponding sides of equal triangles.] Q. E. D.

PROPOSITION V. THEOREM

161. *In the same circle or equal circles, if two arcs are unequal and each is less than half a circumference, the greater arc is subtended by the greater chord; conversely, the greater chord subtends the greater arc.*



GIVEN arc CD greater than arc AB.

TO PROVE chord CD greater than chord AB.

Construct upon the greater arc CD an arc CE equal to arc AB.

Then chord $CE=$ chord AB. § 160

Draw the radii OC, OD, OE.

Angle COE is less than angle COD, being included within it. Ax. 10

Then triangles COE and COD have two sides (the radii) respectively equal, but the included angles unequal.

Therefore chord $CE <$ chord CD. § 91

Substituting AB for CE,

chord $AB <$ chord CD. Q. E. D.

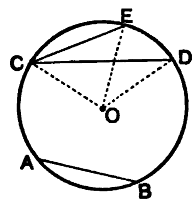
CONVERSELY:

GIVEN chord CD greater than chord AB.

TO PROVE arc CD greater than arc AB.

As before, construct arc CE equal to arc AB.

Then chord $CE=$ chord AB. § 160



But chord $CD >$ chord AB. Hyp.

Substituting CE for AB,

chord $CD >$ chord CE.

Then the triangles COE and COD have two sides respectively equal, but the third sides unequal.

Therefore angle $COE <$ angle COD. § 92

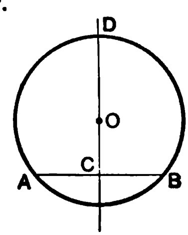
Hence OE, being within the angle COD, must cut off the arc CE less than the arc CD.

Substituting arc AB for arc CE,

arc $AB <$ arc CD. Q. E. D.

PROPOSITION VI. THEOREM

162. *The perpendicular bisector of a chord passes through the centre of the circle.*



GIVEN—circle OAB, chord AB, and CD, the perpendicular bisector of AB.

TO PROVE that CD passes through the centre O.

CD contains all points equally distant from A and B. § 103

[Being the locus of such points.]

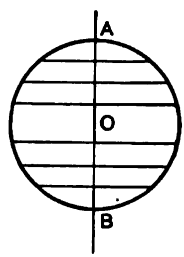
But O is such a point, being the centre.

Therefore CD contains O. Q. E. D.

163. COR. *The diameter perpendicular to a chord bisects it and the subtended arc.*

Hint.—Prove this diameter coincides with the perpendicular bisector. Then draw radii OA and OB, and apply § 159.

164. *Exercise.*—The locus of the middle points of all chords parallel to a given straight line is a diameter perpendicular to the chords.



The student is cautioned in this, and in exercises about loci in general, not to regard the locus found and proved until he has shown *two* things:

(1.) That every point in the proposed locus satisfies the proposed condition, i. e., is the middle point of one of the parallel chords.

(2.) That every point outside of the proposed locus does not satisfy the required condition, i. e., is not the middle point of any of the parallel chords.

Thus the radius is not the locus, being too small (i. e., requirement 1 would be fulfilled, but not 2); and the diameter produced is not, being too large (i. e., requirement 2 would be fulfilled, but not 1).

Some exercises on loci are more easily proved by showing:

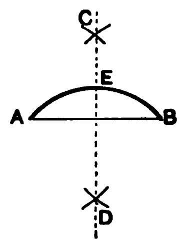
(1.) That every point in the proposed locus satisfies the proposed conditions.

(2.) That every point that satisfies the proposed conditions is in the proposed locus.

The student should show that this method of establishing a locus is equivalent to the previous method.

He may also prove by this method §§ 103 and 106.

165. CONSTRUCTION. *To bisect a given arc:*



GIVEN the arc AEB.

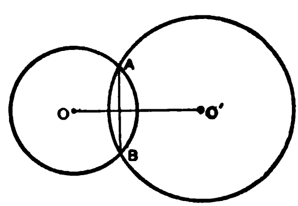
TO CONSTRUCT its bisector.

From A and B as centres, with equal radii greater than a half of AB, describe arcs intersecting at C and D. Draw CD. This line bisects the arc at E.

Hint.—For proof apply § 163.

PROPOSITION VII. THEOREM

166. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



GIVEN two circumferences intersecting at A and B.

TO PROVE—OO' joining their centres is perpendicular to AB at its middle point.

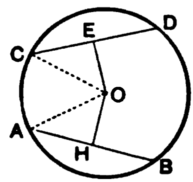
O and O' are each equally distant from A and B. § 146

Therefore OO' bisects AB at right angles. § 104

[Two points equally distant from the extremities of a straight line determine its perpendicular bisector.] Q. E. D.

PROPOSITION VIII. THEOREM

167. *In the same circle or equal circles, equal chords are equally distant from the centre; conversely, chords equally distant from the centre are equal.*



GIVEN CD and AB, equal chords.

TO PROVE—they are at equal distances, EO and HO, from the centre.

Construct radii OC and OA.

E and H are the middle points of CD and AB. § 163

In the right triangles OCE and OAH

$CE=AH$, being halves of equals. Ax. 8

$OC=OA$, being radii.

Hence the triangles are equal. § 101

[Having a side and an hypotenuse respectively equal.]

Therefore $OE=OH$. Q. E. D.

CONVERSELY:

GIVEN $OE=OH$.

TO PROVE $CD=AB$.

In the right triangles OCE and OAH

$OE=OH$. Hyp.

$OC=OA$, being radii.

Hence the triangles are equal. § 101

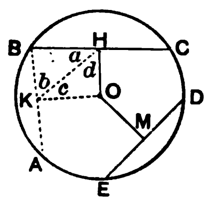
Therefore $CE=AH$.

And $CD=AB$, being doubles of equals. Ax. 7

Q. E. D.

PROPOSITION IX. THEOREM

168. *In the same circle or equal circles, the less of two chords is at the greater distance from the centre; conversely, the chord at the greater distance from the centre is the less.*



GIVEN chord $ED <$ chord BC.

TO PROVE distance $OM >$ distance OH.

Construct from B chord $BA = ED$.
Then its distance $OK = OM$. § 167
And $AB < BC$.

Join K and H.

K and H are the middle points of AB and BC. § 163

Hence $BK < BH$. Ax. 8

[Being halves of unequals.]

Hence angle $a <$ angle b. § 75

[Being opposite unequal sides.]

Subtracting the unequal angles from the equal right angles at H and K,

angle $d >$ angle c. Ax. 6

Therefore $OK > OH$ § 77

[Being opposite unequal angles.]

Substituting OM for OK,

$OM > OH$. Q. E. D.

SUMMARY: $ED < BC$; $BA < BC$; $BK < BH$; $a < b$; $d > c$; $OK > OH$; $OM > OH$.

CONVERSELY:

GIVEN $OM > OH$.

TO PROVE $ED < BC$.

The proof is left to the student.

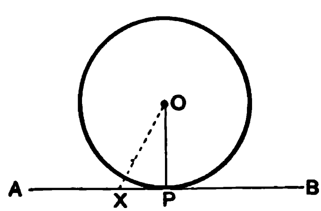
SUMMARY: $OM > OH$; $OK > OH$; $d > c$; $a < b$; $BK < BH$; $BA < BC$; $ED < BC$.

169. *Defs.* — A straight line is **tangent** to a circle if, however far produced, it meets its circumference in but one point.

This point is called the **point of tangency.**

PROPOSITION X. THEOREM

170. *A straight line perpendicular to a radius at its extremity is tangent to the circle; conversely, the tangent at the extremity of a radius is perpendicular to that radius.*



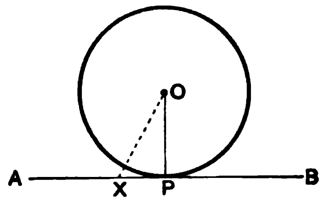
GIVEN—AB perpendicular to the radius OP at its extremity P.

TO PROVE AB is tangent to the circle.

The perpendicular OP is less than any other line OX from O to AB. § 95

[Being the shortest distance from a point to a line.]

Hence, OX being greater than a radius, X lies without the



circumference, and P is the only point in AB on the circumference. Therefore AB is tangent to the circle. Q. E. D.

CONVERSELY:

GIVEN AB tangent to the circle at P.

TO PROVE OP perpendicular to AB.

Since AB is touched only at P, any other point in AB, as X, lies without the circumference.

Hence OX is greater than a radius OP.

Therefore OP, being shorter than any other line from O to AB, is perpendicular to to AB. § 95

Q. E. D.

171. COR. *A perpendicular to a tangent at the point of tangency passes through the centre of the circle.*

Hint.—Suppose a radius to be drawn to the point of tangency.

172. CONSTRUCTION. *At a point P in the circumference of a circle to draw a tangent to the circle.*

Draw the radius OP, and erect PB perpendicular to this radius at P. By § 171 PB is the tangent required.

173. *Exercise.*—The two tangents to a circle from an exterior point are equal.

Hint.—Join the given point and the centre; draw radii to points of tangency.

MEASUREMENT

174. *Def.*—The **ratio** of one quantity to another of the same kind is the number of times the first contains the second.

Thus the ratio of a yard to a foot is three (3), or more fully $\frac{3}{1}$.

175. *Defs.*—To **measure** a quantity is to find its ratio to another quantity of the same kind. The second quantity is called the **unit of measure**; the ratio is called the **numerical measure** of the first quantity.

Thus we measure the length of a rope by finding the number of metres in it; if it contains 6 metres, we say the *numerical measure* of its length is 6, the metre being the *unit of measure.*

176. *Remark.*—If the length of one rope is 20 metres, and the length of another 5 metres, the ratio of their lengths is the number of times 5 metres is contained in 20 metres—that is, the number of times 5 is contained in 20, which is written $\frac{20}{5}$. We may accordingly restate § 175 as follows:

The ratio of two quantities of the same kind is the ratio of their numerical measures.

177. *Defs.*—Two quantities are **commensurable** if there exist a third quantity which is contained a whole number of times in each.

The third quantity is called the **common measure.**

Thus a yard and a mile are commensurable, each containing a foot a whole number of times, the one 3 times, the other 5280 times. Again, a yard and a rod are commensurable. The common measure is not, however, a foot, as a rod contains a foot 16½ times, which is not a whole number of times. But an inch is a common measure, since the yard contains it 36 times and the rod 198 times.

178. *Def.*—Two quantities are **incommensurable** if no third quantity exists which is contained a whole number of times in each.

Thus it can be proved that the circumference and diameter of a circle are incommensurable; also the side and diagonal of a square.

LIMITS

179. *Def.*—A **constant** quantity is one that maintains the same value throughout the same discussion.

180. *Def.*—A **variable** is a quantity which has different successive values during the same discussion.

181. *Def.*—The **limit** of a variable is a constant *from* which the variable can be made to differ by less than any assigned quantity, but *to* which it can never be made equal.

Thus suppose a point P to move over a line from A to B in such a way that in the first second it passes over half the distance, in the next second half the remaining distance, in the third half the new remainder, and so on.

The variable is the *distance* from A to the moving-point. Its successive values are AP', AP'', AP''', etc. If the length of AB is two inches, the value of the variable is first 1 inch, then $1\frac{1}{2}$, $1\frac{3}{4}$, $1\frac{7}{8}$, etc.

(1.) P will *never* reach B, for there is always half of *some* distance remaining.

(2.) P will approach nearer to B than any quantity we may assign.

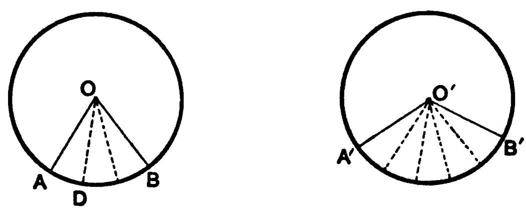
Suppose we assign $\frac{1}{1000}$ of an inch. By continually bisecting the remainder we can reduce it to less than $\frac{1}{1000}$ of an inch. Hence the distance from P to A is a variable whose limit is AB, and the distance from P to B is a variable whose limit is zero.

182. THEOREM. *If two variables approaching limits are always equal, their limits are also equal.*

For two variables that are always equal may be considered as but one variable, and must therefore approach the same limit. Q. E. D.

PROPOSITION XI. THEOREM

183. *In the same circle or equal circles two angles at the centre have the same ratio as their intercepted arcs.*



GIVEN the two equal circles with angles O and O'.

TO PROVE
$$\frac{\text{angle } O'}{\text{angle } O}=\frac{\text{arc } A'B'}{\text{arc } AB}.$$

CASE I. *When the arcs are commensurable.*

Suppose AD is the common measure of the arcs, and is contained three times in AB and five times in $A'B'$.

Then
$$\frac{\text{arc } A'B'}{\text{arc } AB}=\frac{5}{3}. \qquad \S\ 176$$

Draw radii to the several points of division.

The angles O and O' will be subdivided into 3 and 5 parts, *all equal.* § 158

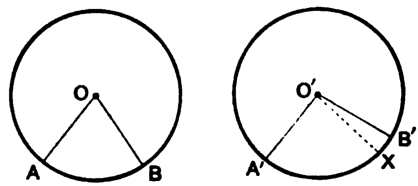
[Being subtended by equal arcs in the same or equal circles.]

Hence
$$\frac{\text{angle } O'}{\text{angle } O}=\frac{5}{3}. \qquad \S\ 176$$

Comparing,
$$\frac{\text{angle } O'}{\text{angle } O}=\frac{\text{arc } A'B'}{\text{arc } AB}. \qquad \text{Ax. } 1$$

Q. E. D.

CASE II. *When the arcs are incommensurable.*



Suppose AB to be divided into any number of equal parts and apply one of these parts to $A'B'$ as a measure as often as it will go.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB' less than one of these parts. § 178

Since AB and $A'X$ are constructed commensurable,

$$\frac{\text{angle } A'O'X}{\text{angle } AOB} = \frac{\text{arc } A'X}{\text{arc } AB}. \qquad \text{Case I}$$

Now suppose the number of parts into which AB is divided to be indefinitely increased.

We can thus make each part as small as we please.

But the remainder, the arc XB', will always be less than one of these parts.

Therefore we can make the arc XB' less than any assigned quantity, though never zero.

Likewise we can make the angle $XO'B'$ less than any assigned quantity, though never zero.

Therefore $A'X$ approaches $A'B'$ as a limit.

Hence $\frac{A'X}{AB}$ approaches $\frac{A'B'}{AB}$ as a limit.

Also angle $A'O'X$ approaches angle $A'O'B'$ as a limit.

Hence $\frac{\text{angle } A'O'X}{\text{angle } AOB}$ approaches $\frac{\text{angle } A'O'B'}{\text{angle } AOB}$ as a limit.

Since the variables $\frac{A'X}{AB}$ and $\frac{\text{angle } A'O'X}{\text{angle } AOB}$ are always equal, so also are their limits.

That is, $$\frac{A'B'}{AB} = \frac{\text{angle } A'O'B'}{\text{angle } AOB}.$$ § 183

Q. E. D.

184. *Exercise.*—In the same circle or equal circles, two sectors have the same ratio as their angles.

The proof is analogous to the preceding, requiring "sector" in place of "arc."

185. *Remark.*—The preceding proposition is often expressed thus:

An angle at the centre *is measured by* its intercepted arc.

This means that if the angle is doubled, the intercepted arc will be doubled; if the angle is halved, the intercepted arc will be halved; if the angle is tripled, the intercepted arc will be tripled; and, in general, if the angle is increased or diminished in any ratio, the intercepted arc will be increased or diminished in the same ratio.

186. *Defs.*—A **degree of angle** is one-ninetieth of a right angle.

A **degree of arc** is the arc intercepted by a degree of angle at the centre.

The arc intercepted by a right angle at the centre is called a **quadrant.**

Hence a quadrant contains 90 degrees (90°) of arc, since a right angle contains 90° of angle.

Also, since four right angles contain 360° of angle, and four right angles intercept a complete circumference, a circumference contains 360° of arc.

Hence a quadrant is one-quarter of a circumference.

187. *Remark.*—These definitions suggest a special form of stating Proposition XI., viz.: The ratio of any angle at the centre to a degree of angle is equal to the ratio of the intercepted arc to the degree of arc, or more briefly: *An angle at the centre contains as many degrees of angle as its intercepted arc contains degrees of arc;* or still again, the numerical measure of an angle at the centre equals the numerical measure of its intercepted arc, the unit of angle being a degree of angle, and the unit of arc being a degree of arc.

The student will be tempted to still further simplify the statement by saying "an angle at the centre *is equal to* its intercepted arc." This, however, is erroneous, because an angle and an arc are not quantities of the same kind, and can no more be called equal than 23 pounds can be said to be equal to 23 yards.

188. *Def.*—An angle is said to be **inscribed** in a circle if its vertex lies in the **circumference** and its sides are chords.

PROPOSITION XII. THEOREM

189. *An inscribed angle is measured by one-half its intercepted arc.**

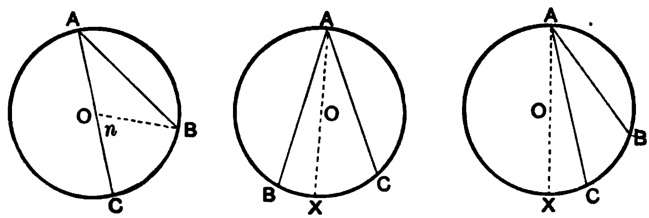


FIG. 1 FIG. 2 FIG. 3

GIVEN the inscribed angle BAC.
TO PROVE—angle BAC is measured by one-half of BC.

CASE I. *When one side AC of the angle is a diameter* (Fig. 1).

Draw the radius OB.

$OA = OB$. § 148

[Being radii.]

Hence angle A = angle B. § 70

[Being base angles of an isosceles triangle.]

* This proposition is first found proved in Euclid (about 300 B.C.), though at least one case, viz., Cor. II., was stated earlier by Thales (about 600 B.C.), the founder of Greek mathematics and philosophy.

But angle $n =$ angle $A +$ angle B. § 58

[The exterior angle of a triangle is equal to the sum of the two opposite interior angles.

Substituting A for B, $n = 2A$.

But n is measured by arc BC. § 185

Hence half of n, or A, is measured by $\frac{1}{2}$ arc BC. Q. E. D.

CASE II. *When the centre O is within the angle* (Fig. 2).

Construct the diameter AX.

Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

Adding, angle BAC is measured by $\frac{1}{2}$ arc $XC + \frac{1}{2}$ arc XB. Ax. 2

Or by $\frac{1}{2}$(arc $XC +$ arc XB).

That is by $\frac{1}{2}$ arc BC.

CASE III. *When the centre is without the angle* (Fig. 3).

Construct the diameter AX.

Angle XAB is measured by $\frac{1}{2}$ arc XB. Case I

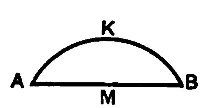
Angle XAC is measured by $\frac{1}{2}$ arc XC. Case I

Substracting, angle BAC is measured by $\frac{1}{2}$ arc BC. Ax. 3

Q. E. D.

190. *Exercise.*—If the inscribed angle is 37° of angle, how many degrees of arc are there in the intercepted arc? How many in the remainder of the circumference? If the intercepted arc is 17°, how large is the inscribed angle?

191. *Defs.*—A **segment** of a circle is the portion of a circle included between an arc and its chord, as $AKBM$.



192. *Def.*—An angle is **inscribed** in a segment of a circle when its vertex is in the arc of the segment and its sides pass through the extremities of that arc.

193. COR. I. *All angles* (A, B, C, Fig. 1) *inscribed in the same segment are equal.*

For they are measured by one-half the same arc MN.

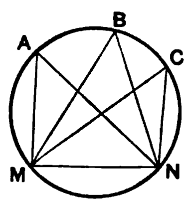


FIG. 1

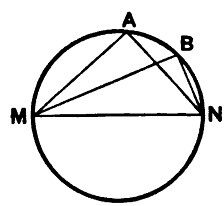


FIG. 2

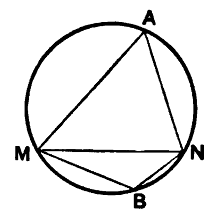


FIG. 3

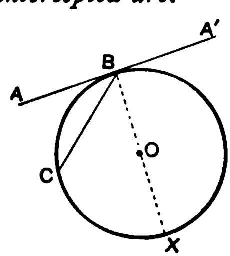
194. COR. II. *An angle* (A, B, Fig. 2) *inscribed in a semicircle is a right angle.*

195. COR. III. *An angle* (A, Fig. 3) *inscribed in a segment greater than a semicircle is an acute angle.*

196. COR. IV. *An angle* (B, Fig. 3) *inscribed in a segment less than a semicircle is an obtuse angle.*

PROPOSITION XIII. THEOREM

197. *An angle formed by a tangent and a chord is measured by one-half its intercepted arc.*



GIVEN—the angle ABC formed by the tangent AB and the chord BC.

TO PROVE—angle ABC is measured by one-half the arc BC.

Construct the diameter BX.

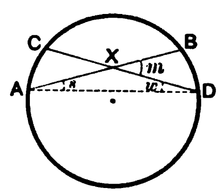
Since a right angle is measured by one-half a semicircumference,

	angle ABX is measured by $\frac{1}{2}$ arc BCX.	
But	angle CBX is measured by $\frac{1}{2}$ arc CX.	§ 189
Subtracting,	angle ABC is measured by $\frac{1}{2}$ arc BC.	Q. E. D.

198. *Exercise.*—An arc contains 16°; at its extremities tangents are drawn. What kind of a triangle do they form with the chord, and how large is each angle?

PROPOSITION XIV. THEOREM

199. ***The angle between two chords which intersect within the circumference is measured by one-half the sum of its intercepted arc and the arc intercepted by its vertical angle.***



GIVEN two intersecting chords AB and CD.

TO PROVE—angle BXD is measured by one-half the sum of the arcs BD and AC.

Join A and D.

Now	$m = s + w$.	§ 58

[An exterior angle of a triangle is equal to the sum of the opposite interior angles.]

But	angle s is measured by $\frac{1}{2}$ arc BD.	§ 189
And	angle w is measured by $\frac{1}{2}$ arc AC.	§ 189
Hence	m is measured by $\frac{1}{2}$ (arc BD + arc AC).	Ax. 2

Q. E. D.

200. *Exercise.*—One angle of two intersecting chords subtends 30° of arc; its vertical angle subtends 40°. How large is the angle? If an angle of two intersecting chords is 15°, and its intercepted arc is 20°, how large is the opposite arc?

201. *Def.*—A **secant** of a circle is a straight line which cuts the circle.

It is therefore a chord produced.

PROPOSITION XV. THEOREM

202. *The angle between two secants intersecting without the circumference, the angles between a tangent and a secant, and the angle between two tangents, are each measured by one-half the difference of the intercepted arcs.*

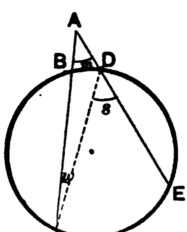


FIG. 1

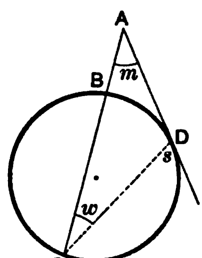


FIG. 2

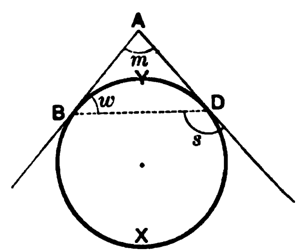


FIG. 3

CASE I. *Two secants* (Fig. 1).

GIVEN two secants, AC and AE.

TO PROVE—angle m is measured by $\frac{1}{2}$ (arc CE—arc BD).

Join C and D.

Then $m + w = s$. § 58

[An exterior angle of a triangle is equal to the sum of the two opposite interior angles.]

Hence $m = s - w$. Ax. 3

But	s is measured by $\frac{1}{2}$ arc CE.	§ 189
And	w is measured by $\frac{1}{2}$ arc BD.	§ 189
Hence	m is measured by $\frac{1}{2}$ (arc CE — arc BD).	Ax. 3

Q. E. D.

CASE II. *A tangent and a secant* (Fig. 2).

GIVEN tangent AD and secant AC.

TO PROVE m is measured by $\frac{1}{2}$ (arc DC — arc BD).

Join C and D.

	$m = s - w$.	§ 58
	s is measured by $\frac{1}{2}$ arc DC.	§ 197
	w is measured by $\frac{1}{2}$ arc BD.	§ 189
Hence	m is measured by $\frac{1}{2}$ (arc DC — arc BD).	Ax. 3

Q. E. D.

CASE III. *Two tangents* (Fig. 3).

	$m = s - w$.	§ 58
	s is measured by $\frac{1}{2}$ arc BXD.	§ 197
	w is measured by $\frac{1}{2}$ arc BYD.	§ 197
Hence	m is measured by $\frac{1}{2}$ (arc BXD — arc BYD).	Ax. 3

Q. E. D.

203. *Exercises.*—In Fig. 1, if CE is 50° and BD is 10°, what is m?

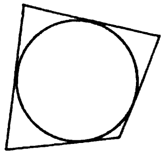
In Fig. 1, if m is 16° and BD is 15°, what is CE?

In Fig. 2, if m is 31° and arc DC is 150°, what is arc BD? and what is arc BC?

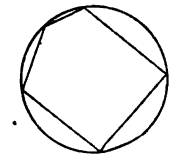
In Fig. 3, if arc BYD is 47°, what is BXD, and what is m?

In Fig. 3, if m is 33°, what are the arcs BXD and BYD?

204. *Def.*—A circle is said to be **inscribed** in a polygon, if it be tangent to every side of the polygon. In the same case, the polygon is said to be **circumscribed** about the circle.



INSCRIBED CIRCLE



CIRCUMSCRIBED CIRCLE

205. *Def.*—A circle is said to be **circumscribed** about a polygon, if the circumference of the circle passes through every vertex of the polygon. In the same case, the polygon is said to be inscribed in the circle.

206. *Exercise.*—To inscribe a circle in a given triangle.

Hint.—Draw the bisectors of two of the angles (Fig. 1).

With O, the intersection of the bisectors, as a centre and the distance to any side as a radius, describe a circumference.

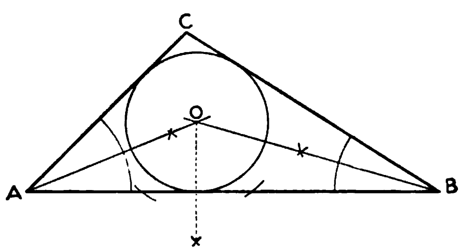


FIG. 1

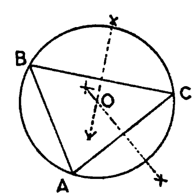


FIG. 2

207. *Exercise.*—To circumscribe a circle about a given triangle.

Hint.—Erect the perpendicular bisectors of two of the sides (Fig. 2).

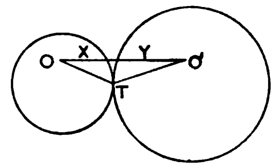
With O, their intersection as a centre, and the distance to any vertex as a radius, describe a circumference.

208. *Remark.*—The construction of the two preceding exercises enables us, first, to draw a circumference tangent to three given lines *not meeting in a point;* second, to draw a circumference through three points *not in the same straight line;* third, to find the centre of a given circumference.

209. *Defs.*—Two circles are **tangent** which touch at but one point. They may be tangent **internally,** so that one circle is within the other; or **externally,** so that each is without the other.

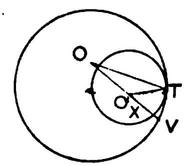
PROBLEMS OF DEMONSTRATION

210. *Exercise.*—The straight line joining the centres of two circles tangent externally passes through the point of tangency.



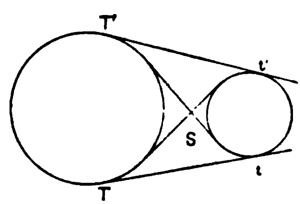
Hint.—Suppose OO' not through T, and prove OO' greater than and also less than the sum of the radii.

211. *Exercise.*—The straight line joining the centres of two circles internally tangent passes through the point of tangency.



Hint.—If not, prove the distance between centres greater than and also less than the difference of the radii.

212. *Defs.*—If each of two circles is entirely without the other, four common tangents can be drawn. Two of these are called external, and two internal. An **external tangent** is one such that the two circles lie on the same side of it; an **internal tangent** is one such that the two circles lie on opposite sides of it.



Question.—In case the two circles are themselves tangent externally, how many common tangents of each kind can be drawn? In case the two circles overlap? In case they are tangent internally?

213. *Exercise.*—The two common external tangents to two circles meet the line joining their centres in the same point. Also the two common internal tangents meet the line of centres in the same point.

214. *Exercise.*—The sum of two opposite sides of a quadrilateral circumscribed about a circle is equal to the sum of the other two sides (§ 173).

215. *Exercise.*—The sum of two opposite angles of a quadrilateral inscribed in a circle is equal to the sum of the other two angles, and is equal to two right angles.

216. *Exercise.*—Two circles are tangent externally at A. The line of centres contains A, by § 210. Prove (1) that the perpendicular to the line of centres at A is a common tangent; (2) that it bisects the other two common tangents; and (3) that it is the locus of all points from which tangents drawn to the two circles are equal.

217. *Exercise.*—Find the locus of the middle points of all chords of a given length.

218. *Exercise.*—If a straight line be drawn through the point of contact of two tangent circles forming chords, the radii drawn to the remaining extremities of these chords are parallel. Also, the tangents at these extremities are parallel. What two cases are possible?

PROBLEMS OF CONSTRUCTION

219. *Exercise.*—Draw a straight line tangent to a given circle and parallel to a given straight line.

220. *Exercise.*—Construct a right triangle, given the hypotenuse and an acute angle.

221. *Exercise.*—Construct a right triangle, given the hypotenuse and a side.

222. *Exercise.*—Construct a right triangle, given the hypotenuse and the distance of the hypotenuse from the vertex of the right angle.

223. *Exercise.*—Construct a circle tangent to a given straight line and having its centre in a given point.

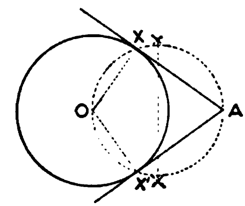
224. *Exercise.*—Construct a circumference having its centre in a given line and passing through two given points.

225. *Exercise.*—Find the locus of the centres of all circles of given radius tangent to a given straight line.

226. *Exercise.*—Construct a circle of given radius tangent to two given straight lines.

227. *Exercise.*—Construct a circle of given radius tangent to two given circles.

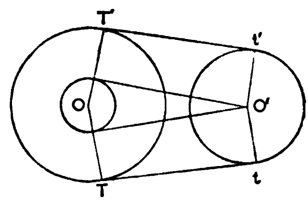
228. *Exercise.* — Draw a tangent to a given circle from a given point without.



Hint.—Upon AO as diameter construct a circumference intersecting the given circumference at X and X'.

Then AX and AX' are the required tangents.

229. *Exercise.* — Construct all the common tangents to two given circles.



Hint.—For the external tangents draw a circle with radius equal to the difference of the radii of the given circles and its centre at the centre of the larger circle. Draw tangents to this circle from the centre of the smaller circle.

PLANE GEOMETRY

BOOK III

PROPORTION AND SIMILAR FIGURES

230. *Def.*—A **proportion** is an equality of ratios.

Thus, if the ratio $\frac{A}{B}$ is equal to the ratio $\frac{C}{D}$, then the equality $\frac{A}{B}=\frac{C}{D}$ constitutes a proportion.

This may also be written

$$A:B=C:D, \text{ or } A:B::C:D,$$

and is read, A is to B as C is to D.

231. *Def.*—The four magnitudes A, B, C, D are called the **terms** of the proportion.

232. *Defs.*—The first and last terms are the **extremes**, the second and third, the **means**.

233. *Defs.*—The first and third terms are called the **antecedents**, and the second and fourth the **consequents**.

234. THEOREM. *If four quantities are in proportion, their numerical measures are in proportion; and conversely.*

GIVEN $$\frac{A}{B}=\frac{C}{D}.$$

TO PROVE: $\frac{a}{b}=\frac{c}{d}$, where a, b, c, d are the numerical measures of A, B, C, D, respectively.

Now $\frac{A}{B}=\frac{a}{b}$ and $\frac{C}{D}=\frac{c}{d}$. § 176

[The ratio of two quantities is the ratio of their numerical measures.]

Whence $\frac{a}{b}=\frac{c}{d}$. Ax. I

Q. E. D.

CONVERSELY: If $\frac{a}{b}=\frac{c}{d}$, then $\frac{A}{B}=\frac{C}{D}$. This can be proved in like manner.

235. *Remark.*—In order that the preceding theorems shall hold true, A and B must be quantities of the *same kind*, as two straight lines, or two angles, and C and D also of the same kind; *but it is not necessary that A and B shall be of the same kind as C and D.*

236. *Def.*—One variable quantity is said to be **proportional** to another when any two values of the first have the same ratio as two corresponding values of the second.

Thus, Proposition XI., Book II., may be expressed:

An angle at the centre of a circle is proportional to its intercepted arc.

By this we mean that the ratio of a given angle, as AOB, to some other angle, as $A'O'B'$, is equal to the ratio of the corresponding arcs, AB and $A'B'$.

TRANSFORMATION OF PROPORTIONS

237. THEOREM. *If four numbers are in proportion, the product of the extremes is equal to the product of the means.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $ad=bc$. (2)

Clear (1) of fractions, i. e., multiply both sides by bd, the product of the denominators of (1).

We have $ad=bc$. (2) Ax. 7

Q. E. D.

238. THEOREM. Conversely, *if the product of two numbers is equal to the product of two others, either pair may be made the extremes and the other pair the means of a proportion.*

GIVEN $ad = bc.$ (2)

TO PROVE $\frac{a}{b} = \frac{c}{d}.$ (1)

Divide both sides of (2) by bd, the product of the denominators of (1).

We have $\frac{a}{b} = \frac{c}{d}.$ (1) Ax. 8

Q. E. D.

Again,

GIVEN $bc = ad.$ (2)

TO PROVE $\frac{b}{a} = \frac{d}{c}.$ (3)

Dividing (2) by ac, the product of the denominators of (3), we obtain (3). Q. E. D.

Question.—By dividing the equation $ad = bc$ by the product of two of the letters, one being from each side, how many proportions in all can be obtained? Write them. If the equation be written $bc = ad$, how many can be obtained, and how do they differ from the former set?

239. *Remark.*—The student has already noticed that the process by which equation (1) was obtained from (2) was the reverse of that by which (2) was obtained from (1). Also it is easy to see that (3) was obtained from (2) by a process the reverse of that by which (2) could have been obtained from (3). Now it is always much easier to see how an equation can be reduced to $ad = bc$ than to see how it can be deduced from $ad = bc$. Since the latter is the reverse of the former, we have the following practical guide for obtaining a required equation from $ad = bc$: First see what processes would be necessary to reduce the equation to $ad = bc$; reverse these steps in order to obtain the method required.

The preceding rule will be better understood from the following example:

240. If $ad = bc$ (2), prove $\frac{a+b}{b} = \frac{c+d}{d}$. (5)

As it is not at first evident what operations to perform on (2) to obtain (5), let us see what would be necessary in the reverse proof. These operations, as the student will easily see, would be:

Step 1.—Clear (5) of fractions, i.e., multiply both sides by bd.

Step 2.—Cancel bd, i.e., subtract bd from both sides.

By the rule of § 240 we need to reverse these steps, viz.:

First, add bd to both sides of (2).

This gives $ad + bd = bc + bd$. Ax. 2

Secondly, divide both sides by bd.

This gives $\frac{a+b}{b} = \frac{c+d}{d}$. (5) Ax. 8

241. THEOREM. *If four numbers are in proportion, they are also in proportion by inversion.*

GIVEN $\frac{a}{b} = \frac{c}{d}$. (1)

TO PROVE $\frac{b}{a} = \frac{d}{c}$. (3)

OUTLINE PROOF.—Derive from (1) equation (2), or $bc = ad$, and from (2) equation (3) by the rule of § 240.

242. *Exercise.*—Prove § 241 otherwise.

243. THEOREM. *If four numbers are in proportion, they are also in proportion by alternation.*

GIVEN $\frac{a}{b} = \frac{c}{d}$. (1)

TO PROVE $\frac{a}{c} = \frac{b}{d}$. (4)

Hint.—Proceed as in § 242 or multiply each side of (1) by $\frac{b}{c}$.

244. THEOREM. *If four numbers are in proportion, they are also in proportion by composition.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a+b}{b}=\frac{c+d}{d} \qquad (5)$$

Hint.—Proceed as in § 242 or add 1 to each side of equation (1).

245. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{a}=\frac{c+d}{c}$.

246. THEOREM. *If four numbers are in proportion, they are also in proportion by division.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a-b}{b}=\frac{c-d}{d}. \qquad (6)$$

Hint.—Proceed as in § 242 or subtract 1 from each side of equation (1).

247. *Exercise.*—If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a-b}{a}=\frac{c-d}{c}$.

248. THEOREM. *If four numbers are in proportion, they are also in proportion by composition and division.*

GIVEN $$\frac{a}{b}=\frac{c}{d}. \qquad (1)$$

TO PROVE $$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad (7)$$

Hint.—Divide equation (5) by (6), or proceed as in § 242.

249. THEOREM. *If four numbers are in proportion, equimultiples of the antecedents will be in proportion with equimultiples of the consequents.*

GIVEN $\frac{a}{b}=\frac{c}{d}$. (1)

TO PROVE $\frac{ma}{nb}=\frac{mc}{nd}$. (8)

Hint.—This is proved by multiplying each side of (1) by $\frac{m}{n}$.

250. *Remark.*—The equations so far considered are

$$\frac{a}{b}=\frac{c}{d} \quad (1)$$

$$ad=bc \quad (2)$$

$$\frac{b}{a}=\frac{d}{c} \quad (3)$$

$$\frac{a}{c}=\frac{b}{d} \quad (4)$$

$$\frac{a+b}{b}=\frac{c+d}{d} \quad (5)$$

$$\frac{a-b}{b}=\frac{c-d}{d} \quad (6)$$

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \quad (7)$$

$$\frac{ma}{nb}=\frac{mc}{nd}. \quad (8)$$

The student will see that, if any one of these equations be given, all the others can be obtained. For the given equation can be transformed into (2), and (2) into any other by the method of § 239.

251. THEOREM. *In a series of equal ratios the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

GIVEN $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{h}{k}=$ etc.

TO PROVE $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=$ etc.

Call each one of the equal ratios $\frac{a}{b}$, $\frac{c}{d}$, etc., r.

Then $\frac{a}{b}=r$, or $a=br$. Ax. 7

$\frac{c}{d}=r$, or $c=dr$.

$\frac{e}{f}=r$, or $e=fr$.

Adding these equations together,

$$a+c+e=br+dr+fr=r(b+d+f). \quad \text{Ax. 2}$$

Dividing both sides by $b+d+f$ gives

$$\frac{a+c+e}{b+d+f}=r. \quad \text{Ax. 8}$$

But $r=\frac{a}{b}=\frac{c}{d}=$ etc.

Therefore $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=$ etc. Ax. 1

Q. E. D.

252. *Exercise.*—If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, etc.

Prove that $\frac{a}{c}=\frac{a^2}{b^2}$, $\frac{a}{d}=\frac{a^3}{b^3}$, $\frac{a}{e}=\frac{a^4}{b^4}$, etc.

Hint.—To prove that $\frac{a}{c}=\frac{a^2}{b^2}$, multiply each side of $\frac{a}{b}=\frac{b}{c}$ by $\frac{a}{b}$.

To prove that $\frac{a}{d}=\frac{a^3}{b^3}$, multiply the left side of $\frac{a}{c}=\frac{a^2}{b^2}$ by $\frac{c}{d}$, the right side by its equal $\frac{a}{b}$.

253. THEOREM. *The products of the corresponding terms of any number of proportions from a proportion.*

GIVEN
$$\begin{cases} \dfrac{a}{b}=\dfrac{c}{d}, \\ \dfrac{a'}{b'}=\dfrac{c'}{d'}, \\ \dfrac{a''}{b''}=\dfrac{c''}{d''}, \\ \text{etc.} \end{cases}$$

TO PROVE
$$\frac{aa'a''}{bb'b''}=\frac{cc'c''}{dd'd''}.$$

Multiply all the given equations together.

The result is
$$\frac{aa'a''}{bb'b''}=\frac{cc'c''}{dd'd''}.$$
Q. E. D.

254. THEOREM. *If four numbers are in proportion, like powers of these numbers are in proportion.*

GIVEN
$$\frac{a}{b}=\frac{c}{d}.$$

TO PROVE
$$\frac{a^2}{b^2}=\frac{c^2}{d^2};\ \frac{a^3}{b^3}=\frac{c^3}{d^3};\ \frac{a^4}{b^4}=\frac{c^4}{d^4};\ \text{etc.}$$

This is proved by raising the two sides of the given equation to the required power.

255. *Def.*—The **segments** of a straight line are the parts into which it is divided.

256. *Def.*—Two straight lines are **divided proportionally,** when the ratio of one line to either of its segments is equal to the ratio of the other line to its corresponding segment.

PROPOSITION I. THEOREM

257. *A straight line parallel to one side of a triangle divides the other two sides proportionally.*

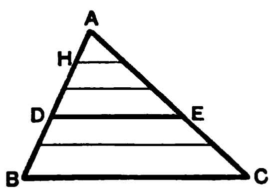


FIG. 1

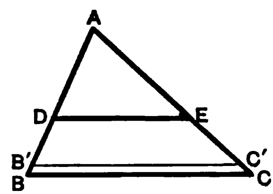


FIG. 2

GIVEN—the straight line DE parallel to the side BC of the triangle ABC.

TO PROVE $\frac{AB}{AD}=\frac{AC}{AE}$.

CASE I.—*When AB and AD are commensurable* (Fig. 1).

Let AH be the unit of measure, and suppose it is contained in AB five times, and in AD three times.

Then $\frac{AB}{AD}=\frac{5}{3}$. (1) § 176

Through the several points of division on AB and AD draw lines parallel to BC.

These lines will divide AC into five equal parts, of which AE contains three. § 124

[If any number of parallels intercept equal parts on one cutting line, they will intercept equal parts on every other cutting line.]

Therefore $\frac{AC}{AE}=\frac{5}{3}$. (2) § 176

Comparing (1) and (2), $\frac{AB}{AD}=\frac{AC}{AE}$. Ax. 1

Q. E. D.

CASE II.—*When AB and AD are incommensurable* (Fig. 2).

Let AD be divided into any number of equal parts, and let one of these parts be applied to AB as a measure.

Since AD and AB are incommensurable, a certain number of these parts will extend from A to B', leaving a remainder BB' less than one of these parts.

Through B' draw $B'C'$ parallel to BC.

Since AD and AB' are commensurable,

$$\frac{AB'}{AD}=\frac{AC'}{AE}. \qquad \text{Case I}$$

Now, suppose the number of divisions of AD to be indefinitely increased.

Then each division, either of AD or of AE, can be made as small as we please.

Hence $B'B$ and $C'C$, being always less than one of these divisions, can be made as small as we please.

Hence $\left.\begin{array}{l} AB' \text{ approaches } AB \text{ as a limit.} \\ AC' \text{ approaches } AC \text{ as a limit.}\end{array}\right\}$ § 181

Hence $\left.\begin{array}{l} \dfrac{AB'}{AD} \text{ approaches } \dfrac{AB}{AD} \text{ as a limit.} \\ \dfrac{AC'}{AE} \text{ approaches } \dfrac{AC}{AE} \text{ as a limit.}\end{array}\right\}$

But we proved $\dfrac{AB'}{AD}=\dfrac{AC'}{AE}.$

Hence $\dfrac{AB}{AD}=\dfrac{AC}{AE}.$ § 182

Q. E. D.

258. COR. I. $\dfrac{AD}{DB}=\dfrac{AE}{EC}.$

Hint.—This is proved by division and inversion.

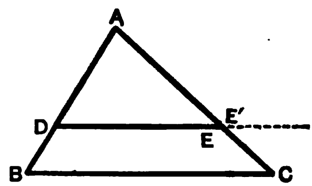
259. COR. II. $\dfrac{AB}{AC}=\dfrac{AD}{AE}=\dfrac{DB}{EC}.$

Hint—This is proved by alternation.

PROPOSITION II. THEOREM

260. *If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.*

[Converse of Proposition I.]



GIVEN—the straight line DE, in the triangle ABC, so drawn that

$$\frac{AB}{AD}=\frac{AC}{AE}.$$

TO PROVE DE parallel to BC.

From D draw DE' parallel to BC.

Then $$\frac{AB}{AD}=\frac{AC}{AE'}.$$ § 257

[A straight line parallel to one side of a triangle divides the other two sides proportionally.]

But $$\frac{AB}{AD}=\frac{AC}{AE}.$$ Hyp.

Hence $$\frac{AC}{AE}=\frac{AC}{AE'}.$$ Ax. 1

The numerators of these equal fractions being equal, their denominators must also be equal. § 241, Ax. 7

That is, $AE=AE'$.

Therefore E and E' coincide.

Hence DE and DE' coincide. Ax. *a*

But DE' is parallel to BC by construction.

Therefore DE, which coincides with DE', is parallel to BC.

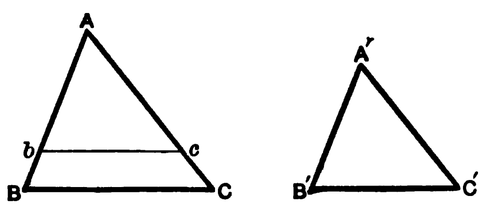
Q. E. D.

261. *Def.*—**Similar polygons** are polygons which have the angles of one equal to the angles of the other, each to each, and the corresponding, or **homologous,** sides proportional.*

It will be shown later that if the polygons are triangles, neither of these two conditions can be true without the other ; but if the polygons have four or more sides, either can be true without the other.

PROPOSITION III. THEOREM

262. *Two triangles which are mutually equiangular are similar.*



GIVEN — in the triangles ABC and $A'B'C'$, the angles A, B, and C, equal respectively to the angles A', B', C'.

TO PROVE the triangle ABC similar to $A'B'C'$.

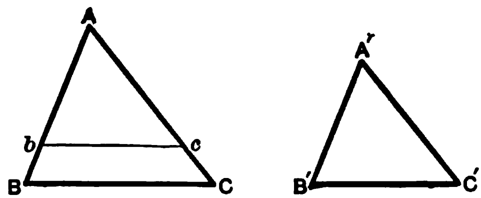
Apply the triangle $A'B'C'$ to ABC so that the angle A' shall fall on A.

Then the triangle $A'B'C'$ will take the position Abc.

Since the angle Abc (or the angle B') is given equal to B, bc is parallel to BC. § 44

[If two straight lines are cut by a third, so that corresponding angles are equal, the straight lines are parallel.]

* There is some evidence that the early Egyptians knew of the properties of similar figures. But the first philosopher who is mentioned as employing them is Thales (600 B.C.). One of his simplest calculations was to find the height of a building by measuring its shadow at that hour of the day when a man's shadow is of the same length as himself.



Hence $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ § 257

or $$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

By applying the triangle $A'B'C'$ to ABC so that angle B' shall coincide with its equal B, it may be shown in the same manner that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

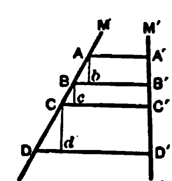
Therefore $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Hence the homologous sides are proportional and the triangles are similar. § 261

Q. E. D.

263. COR. I. *If two triangles have two angles of the one equal to two angles of the other, the triangles are similar.*

264. COR. II. *If two straight lines are cut by a series of parallels, the corresponding segments of the two lines are proportional.*

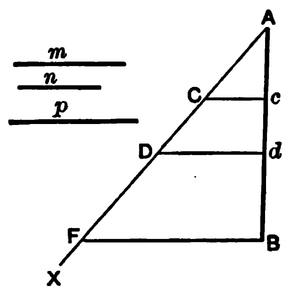


Hint.—Let MN and $M'N'$ be cut by the parallels AA', BB', CC', and DD'.

Draw Ab, Bc, and Cd parallel to $M'N'$.

Prove the triangles ABb, BCc, and CDd similar.

265. CONSTRUCTION. *To divide a given straight line into parts proportional to given straight lines.*



Required.—To divide AB into parts proportional to m, n, and p.

From A draw an indefinite straight line AX, upon which lay off $AC=m$, $CD=n$, and $DF=p$.

Join F and B, and draw Dd and Cc parallel to FB.

Ac, cd, and dB will then be proportioned to m, n, and p. § 264

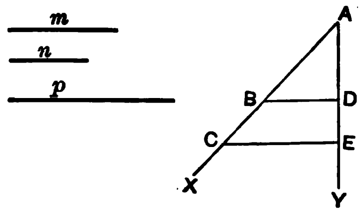
Q. E. F.

266. *Remark.*—If the lines m, n, and p are equal to each other, the line AB will be divided into equal parts. (See also § 124.)

267. *Def.*—A **fourth proportional** to three given quantities is the fourth term of a proportion whose first three terms are the three given quantities taken in order.

268. *Defs.*—When the two means of a proportion are equal, either of them is said to be a **mean proportional** between the other two terms. The fourth term in this case is called a **third proportional** to the other two.

269. CONSTRUCTION. *To find a fourth proportional to three given straight lines.*



Required.—To find a fourth proportional to m, n, and p.

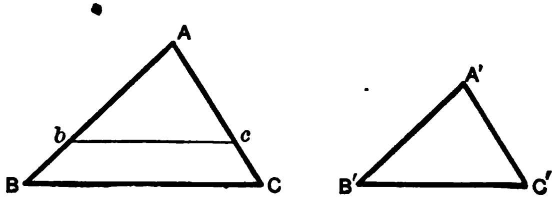
Draw from A the two indefinite lines AX and AY.
Lay off $AB=m$, $AD=n$, and $AC=p$.
Join B and D, and through C draw CE parallel to BD.
Then AE will be the fourth proportional.

For $$\frac{AB}{AD}=\frac{AC}{(AE)}. \qquad \S\ 257$$

270. *Remark.*—If n and p are equal, then also AC and AD are equal, and AE is a third proportional to AB and AD.

PROPOSITION IV. THEOREM

271. *Two triangles are similar when their homologous sides are proportional.*



GIVEN—in the two triangles ABC and $A'B'C'$,

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

TO PROVE the triangle ABC similar to $A'B'C'$.

On AB lay off $Ab=A'B'$, and on AC lay off $Ac=A'C'$, and join bc.

Then by substituting Ab and Ac for their equals $A'B'$ and $A'C'$ in the given proportion, we have

$$\frac{AB}{Ab}=\frac{AC}{Ac}.$$

Therefore the line bc is parallel to BC. § 260

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

And the angle Abc = the angle B, and $Acb=C$. § 48

Hence the triangles ABC and Abc, being mutually equiangular, are similar. § 262

It remains to show that the triangle Abc equals the triangle $A'B'C'$. Since two of their sides are given equal, we only need to show that the third sides bc and $B'C'$ are equal.

Now $$\frac{bc}{BC}=\frac{Ab}{AB}=\frac{A'B'}{AB}.$$ § 261

But $$\frac{B'C'}{BC}=\frac{A'B'}{AB}.$$ Hyp.

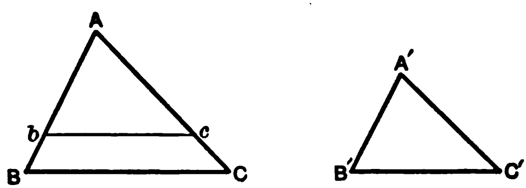
Hence $$\frac{bc}{BC}=\frac{B'C'}{BC}.$$ Ax. 1

Hence $$bc=B'C'.$$ Ax. 7

Therefore the triangles Abc and $A'B'C'$ are equal. § 88
But the triangle Abc has been proved similar to ABC.
Hence $A'B'C'$, the equal of Abc, is similar to ABC. Q. E. D.

PROPOSITION V. THEOREM

272. *Two triangles are similar when an angle of the one is equal to an angle of the other, and the sides including these angles are proportional.*



GIVEN—in the triangles ABC and $A'B'C'$, the angle $A=A'$ and

$$\frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

TO PROVE the triangles similar.

Place the triangle $A'B'C'$ on ABC so that the angle A' shall coincide with A, and B' fall at b, and C' at c.

Then $$\frac{AB}{Ab}=\frac{AC}{Ac}.$$ Hyp.

Therefore bc is parallel to BC, § 260

[If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.]

and the angles b and c are equal respectively to B and C. § 48

Hence the triangles ABC and Abc are similar. § 262

[Two triangles which are mutually equiangular are similar.]

But Abc is equal to $A'B'C'$.

Therefore the triangle $A'B'C'$ is also similar to ABC.

Q. E. D.

PROPOSITION VI. THEOREM

273. *Two triangles which have their sides parallel each to each, or perpendicular each to each, are similar.*

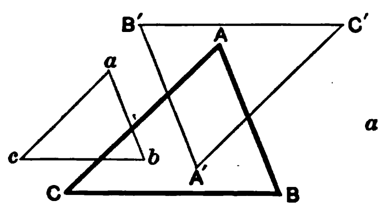


FIG. 1

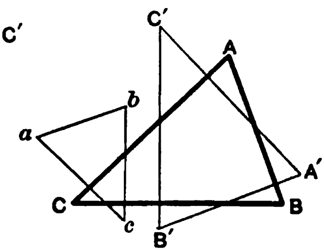


FIG. 2

GIVEN—in the triangles $A'B'C'$ and ABC, that the sides $A'B'$, $A'C'$, and $B'C'$ are respectively parallel to AB, AC, and BC in Fig. 1, and perpendicular in Fig. 2.

TO PROVE the triangles similar.

Since the sides of the two triangles in Fig. 1 are parallel, and in Fig. 2 are perpendicular each to each, the included angles formed by each pair of sides are in both cases either equal or supplementary. §§ 50, 52

Hence, in both cases, we can make three hypotheses, as follows:

1st hypothesis, $A+A'=2$ right angles; $B+B'=2$ right angles; $C+C'=2$ right angles.

2d hypothesis, $A=A'$; $B+B'=2$ right angles; $C+C'=2$ right angles.

3d hypothesis, $A=A'$; $B=B'$; and hence also $C=C'$. § 60

Neither the first nor the second of these hypotheses can be true, for then the sum of the angles of a triangle would be more than two right angles.

Therefore the third is the only one admissible.

Hence the two triangles are similar. Q. E. D.

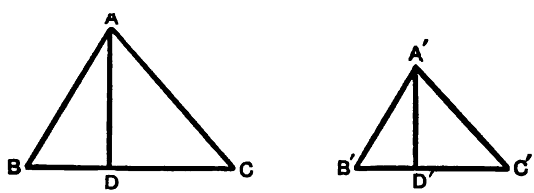
274. *Remark.*—The student will observe that ABC and abc can be proved similar in the same manner.

275. *Remark.* — The homologous sides of the two triangles are any two parallel sides (Fig. 1) or any two perpendicular sides (Fig. 2).

276.—*Defs.*—The **base** of a triangle is that side upon which the triangle is supposed to stand. The **altitude** is the perpendicular to the base from the opposite vertex.

PROPOSITION VII. THEOREM

277. *In two similar triangles, corresponding altitudes have the same ratio as any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, AD and $A'D'$ being their corresponding altitudes.

TO PROVE $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$

The two right triangles ABD and $A'B'D'$ are similar, since B and B' are equal angles, and ADB and $A'D'B'$ are both right angles. § 263

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

Then $$\frac{AD}{A'D'}=\frac{AB}{A'B'}.$$ § 261

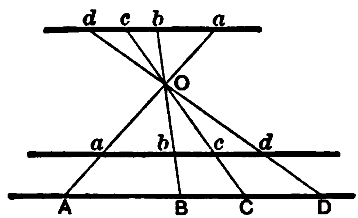
But, since the triangles ABC and $A'B'C'$ are similar, we have $$\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ § 261

Hence $$\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}.$$ Ax. 1

Q. E. D.

PROPOSITION VIII. THEOREM

278. *If three or more straight lines drawn through a common point intersect two parallels, the corresponding segments of the parallels are proportional.*



GIVEN—the lines OA, OB, OC, OD, drawn through a common point O and intersecting the parallels AD and ad in the points A, B, C, D, and a, b, c, d.

TO PROVE $$\frac{AB}{ab}=\frac{BC}{bc}=\frac{CD}{cd}.$$

Since ad is parallel to AD,
angle Oab = angle OAB, and angle Oba = angle OBA. §§ 47, 48

Therefore the triangle aOb is similar to AOB. § 263

[If two triangles have two angles of one equal to two angles of the other, they are similar.]

In the same way the triangles bOc and cOd are similar respectively to BOC and COD.

Therefore $$\frac{ab}{AB}=\left(\frac{Ob}{OB}\right)=\frac{bc}{BC}=\left(\frac{Oc}{OC}\right)=\frac{cd}{CD}.$$ § 261

Whence $$\frac{ab}{AB}=\frac{bc}{BC}=\frac{cd}{CD}.$$ Ax. 1

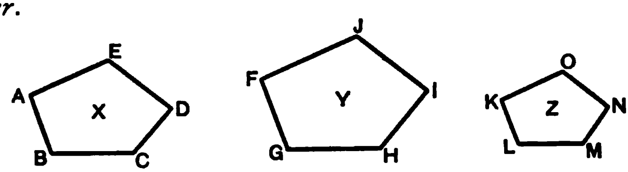
Q. E. D.

279. COR. *If $AB=BC=CD$, then $ab=bc=cd$. Therefore the lines drawn from the vertex of a triangle dividing the base into equal parts divide a parallel to the base into equal parts also.*

280. *Exercise.*—Two men, on opposite sides of a street, walk in opposite directions, and so that a tree between them always hides each from the other. Prove that, if one man walks uniformly, the other must also, and show the connection between the position of the tree and the ratio of their speeds.

PROPOSITION IX. THEOREM

281. *Two polygons similar to a third are similar to each other.*



GIVEN the polygons X and Y, both similar to Z.

TO PROVE that X and Y are similar to each other.

Angles A and F are each equal to K. Hyp.

Therefore they are equal to each other. Ax. 1

In like manner the angles B, C, D, E of X are equal to the corresponding angles of G, H, I, J of Y.

$$\left.\begin{aligned} &\text{Again} \quad \frac{AB}{KL}=\frac{BC}{LM}=\frac{CD}{MN}=\text{etc.,} \\ &\text{and} \quad \frac{FG}{KL}=\frac{GH}{LM}=\frac{HI}{MN}=\text{etc.} \end{aligned}\right\} \quad \S\ 261$$

Dividing the first set of equations by the second,

$$\frac{AB}{FG}=\frac{BC}{GH}=\frac{CD}{HI}=\text{etc.}$$

Therefore X and Y are similar.

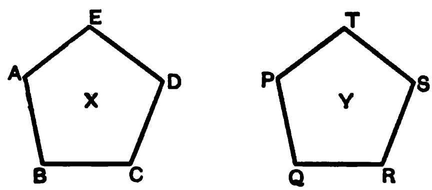
[Having their angles respectively equal and their homologous sides proportional.] Q. E. D.

282. *Def.*—The **ratio of similitude** of any two similar polygons is the ratio of any two homologous sides.

[Thus in § 282 the ratio of AB to FG is the ratio of similitude of X and Y.]

PROPOSITION X. THEOREM

283. *Two similar polygons are equal if their ratio of similitude is unity.*



GIVEN—the similar polygons X and Y, whose ratio of similitude is unity.

TO PROVE X and Y equal.

The angles of X and Y are respectively equal. § 261

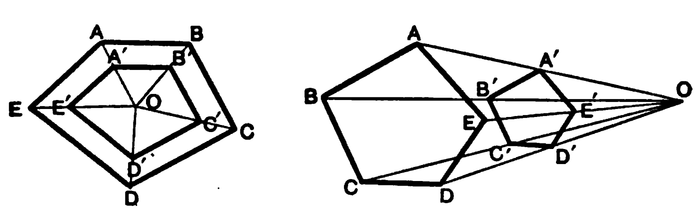
Again $\frac{AB}{PQ}=1.$ Hyp.

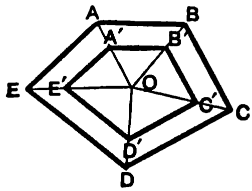
Therefore $AB=PQ$; likewise $BC=QR$; etc.

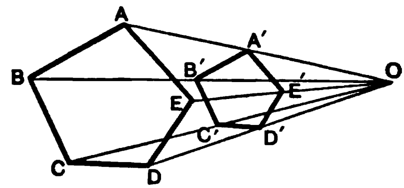
That is, the sides of X and Y are respectively equal.

Hence the polygons, having their corresponding angles and sides respectively equal, can be made to coincide and are equal. Q. E. D.

284. *Defs.*—If the vertices A, B, C, D, etc., of a polygon are joined by straight lines to a point O, and the lines OA, OB, OC, OD, etc., are divided in a given ratio at the points







A', B', C', D', etc., the polygon $A'B'C'D'$, etc., is said to be **radially situated** with respect to the polygon $ABCD$, etc.

The ratio of the lines OA' and OA is called the **ray ratio** of the two polygons.

The point O is called the **ray centre.**

In each of the figures the vertices A and A', B and B', C and C', etc., lie on the rays OA, OB, OC, etc., making

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \text{etc.}$$

The two polygons $ABCDE$ and $A'B'C'D'E$ are therefore radially situated.

The points A', B', C', D' are **homologous** to the points A, B, C, D respectively.

Straight lines determined by homologous points are **homologous.**

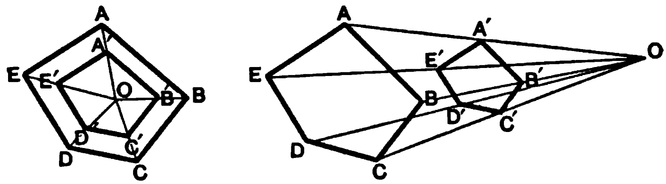
Angles formed by homologous lines are **homologous.**

PROPOSITION XI. THEOREM

285. *Two polygons radially situated are similar, and their ratio of similitude is equal to the ray ratio.*

GIVEN — the polygons $ABCDE$ and $A'B'C'D'E'$ radially situated, O being the ray centre.

TO PROVE—they are similar, and that the ray ratio is their ratio of similitude.



AB is parallel to $A'B'$, BC to $B'C'$, etc. § 260

[If a straight line divide two sides of a triangle proportionally, it is parallel to the third side.]

Hence angle $ABC = A'B'C'$, angle $BCD = B'C'D'$, etc. § 50

[Having their sides parallel, right to right and left to left.]

Again, triangle OAB is similar to $OA'B'$, OBC to $OB'C'$, etc. § 272

Therefore $\frac{AB}{A'B'} = \left(\frac{OB}{OB'}\right) = \frac{BC}{B'C'} = \left(\frac{OC}{OC'}\right) = \text{etc.}$ § 261

Whence $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.}$ Ax. 1

Since the polygons have their angles respectively equal and their homologous sides proportional, they are similar. § 261

Also, their ratio of similitude $\frac{AB}{A'B'} = \text{ray ratio } \frac{OB}{OB'}$. Q. E. D.

286. *Def.*—A **diagonal** of a polygon is a straight line joining two vertices not in the same side.

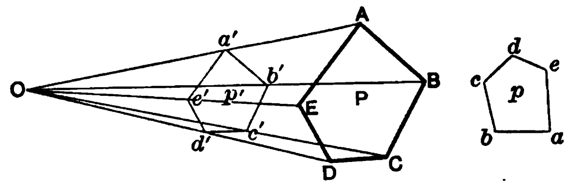
287. *Exercise.*—If two radially situated polygons are divided into triangles by diagonals drawn from corresponding vertices A, A', the corresponding triangles thus formed will also be radially situated.

288. *Def.*—The ray centre is also called the **centre of similitude.**

8

PROPOSITION XII. THEOREM

289. *Any two similar polygons can be radially placed, the ray ratio being equal to the ratio of similitude.*



GIVEN the similar polygons P and p.

TO PROVE—that they can be radially placed, the ray ratio being the ratio of similitude.

With any point O as ray centre form a polygon p' radially situated with regard to P, having the ray ratio $\frac{Oa'}{OA}$ equal to the ratio of similitude $\frac{ab}{AB}$ of p and P.

Then p' and P will be similar, the ratio of similitude being

$$\frac{a'b'}{AB} = \frac{Oa'}{OA}. \qquad \S\ 285$$

But p and P are given similar, and their ratio of similitude is

$$\frac{ab}{AB}.$$

Therefore p' and p are similar. § 281

Now, since $\frac{a'b'}{AB} = \frac{Oa'}{OA}$ and $\frac{Oa'}{OA} = \frac{ab}{AB}$,

$$\frac{a'b'}{AB} = \frac{ab}{AB}. \qquad \text{Ax. 1}$$

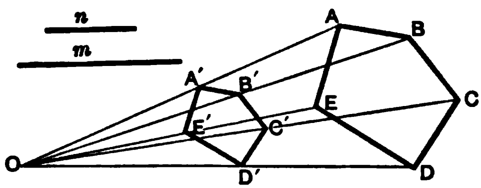
By alternation $\frac{a'b'}{ab} = \frac{AB}{AB} = 1.$ § 243

That is, the ratio of similitude of p' and p is unity.

Therefore p can be made to coincide with p'. § 283

In other words, P and p can be radially placed, the ray ratio being the ratio of similitude. Q. E. D.

290. CONSTRUCTION. ***To construct a polygon similar to a given polygon, having given the ratio of similitude.***



GIVEN the polygon $ABCDE$.

TO CONSTRUCT—similar to $ABCDE$, a polygon $A'B'C'D'E'$, the ratio of similitude being $\frac{m}{n}$.

From any point O draw lines to all the vertices, A, B, C, D, E.

Construct OA' a fourth proportional to m, n, and OA; OB', a fourth proportional to m, n, and OB, etc. § 269

That is $$\frac{m}{n} = \frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \text{etc.}$$

Then the polygons $ABCDE$ and $A'B'C'D'E'$ will be similar, and their ratio of similitude will be $\frac{m}{n}$. § 285

Q. E. F.

291. *Exercise*—To construct a polygon similar to a given polygon, having a given line as a side homologous to a given side of the given polygon.

Hint.—Find the ratio of similitude. Then by § 290 construct a polygon similar to the given polygon having this ratio of similitude. Lastly, upon the given line as a side draw a polygon having its angles and sides equal to those of the second polygon.

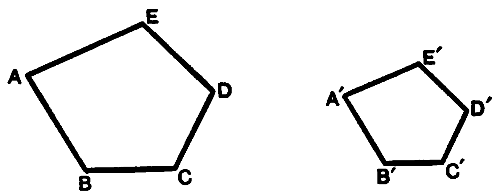
292. *Exercise.*—In two similar polygons, homologous diagonals have the same ratio as any two homologous sides.

Hint.—Place the polygons in a radial position.

293. *Def.*—The **perimeter** of a polygon is the sum of its sides.

PROPOSITION XIII. THEOREM

294. *The perimeters of two similar polygons have the same ratio as any two homologous sides.*



GIVEN—the perimeters P and P' of the two polygons $ABCDE$ and $A'B'C'D'E'$.

TO PROVE $\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=$ etc.

Since the two polygons are similar,

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.} \qquad \S\ 261$$

Then $$\frac{AB+BC+CD+\text{etc.}}{A'B'+B'C'+C'D'+\text{etc.}}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\text{etc.} \qquad \S\ 251$$

That is, $$\frac{P}{P'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}=\text{etc.}$$

Q. E. D.

295. *Remark.*—A pantograph* is a machine for drawing a plane figure similar to a given plane figure.

The pantograph, shown in Figs. 1 and 2, consists of four bars, parallel in pairs and jointed at B, C, D, and E. At D and F are pencils and A turns upon a fixed pivot. BD and DE may be so adjusted as to form a parallelogram $BCED$ cutting AC and CF in any required ratio $\frac{AB}{AC}=\frac{CE}{CF}$.

* The pantograph was invented in 1603 by Christopher Scheiner. It is very useful for enlarging and reducing maps and drawings.

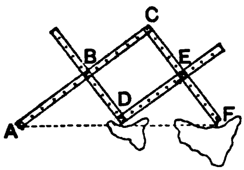


FIG. 1

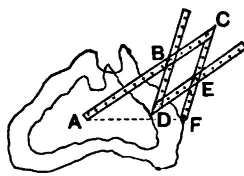


FIG. 2

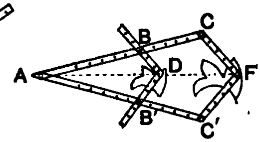


FIG. 3

Then (see § 296) D will always be in the same straight line with A and F and the ratio $\frac{AD}{AF}$ will remain constant and equal to the given ratio $\frac{AB}{AC}$.

Hence, if the pencil F traces a given figure, the pencil D will trace a similar figure, the ratio of similitude being the fixed ratio $\frac{AD}{AF}$.

In Fig. 3 the principle is similar; as also in Fig. 4, where the two figures are on opposite sides of A.

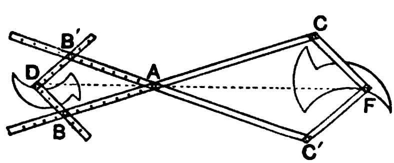


FIG. 4

296. *Exercise.* — Prove the principles stated in § 295, viz., that A, D, F remain always in the same straight line, and that $\frac{AD}{AF}$ remains constant and equal to $\frac{AB}{AC}$.

Hint.—In $\frac{AB}{AC}=\frac{CE}{CF}$ substitute BD for CE and prove the triangles ABD and ACF similar.

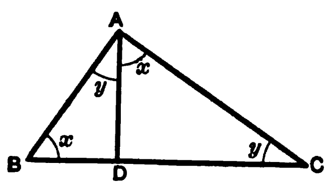
PROPOSITION XIV. THEOREM

297. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

I. ***The triangles on each side of the perpendicular are similar to the whole triangle and to each other.***

II. ***The perpendicular is a mean proportional between the segments of the hypotenuse.***

III. ***Each side about the right angle is a mean proportional between the hypotenuse and the adjacent segment.***



GIVEN—the right triangle ABC and the perpendicular AD from the vertex of the right angle A on BC.

I. TO PROVE—the triangles DBA, DAC, and ABC similar to each other.

The right triangles DBA and ABC each have the angle B common; hence they are mutually equiangular § 60

Also, the right triangles DAC and ABC, having the angle C common, are mutually equiangular. § 60

Hence the three triangles DBA, DAC, and ABC are mutually equiangular.

They are therefore similar. § 262

Q. E. D.

NOTE.—The angles thus proved equal are $B = DAC$, both of which are marked x, and $C = DAB$, both marked y.

II. TO PROVE—AD a mean proportional between DC and BD.

Since the two right triangles DBA and DAC are similar, their homologous sides (that is, the sides opposite equal angles) are proportional. § 261

Hence BD, opposite y in triangle DBA : AD, opposite y in DAC : : AD, opposite x in first : DC opposite x in second.

That is, AD is a mean proportional between BD and DC. § 268

Q. E. D.

III. TO PROVE—AB a mean proportional between BC and BD.

In the similar triangles ABC and DBA.

BC, opposite right angle in the large triangle : BA, opposite right angle in small : : BA, opposite y in first : BD, opposite y in second. § 261

That is, BA is a mean proportional between BC and BD.

In like manner it may be shown that AC is a mean proportional between BC and DC. Q. E. D.

298. COR. I. From II. of the preceding proposition

we have $\overline{AD}^2 = BD \times DC$, (1) § 237

and from III., $\overline{BA}^2 = BC \times BD$, (2)

and $\overline{AC}^2 = BC \times DC$. (3)

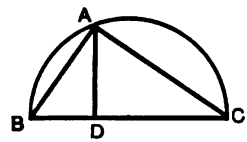
299. COR. II. Dividing (2) by (3)

$$\frac{\overline{BA}^2}{\overline{AC}^2} = \frac{BD}{DC}.$$

Hence, *in a right triangle, the squares of the sides about the right angle are proportional to the segments of the hypotenuse made by a perpendicular let fall from the vertex of the right angle.*

300. *Remark.*—By $\overline{AD}^2$ is understood the square of the numerical measure of AD.

301. COR. III. *If from a point A in the circumference of a circle chords AB and AC be drawn to the extremities of a diameter BC, and AD be drawn from A perpendicular to BC,*

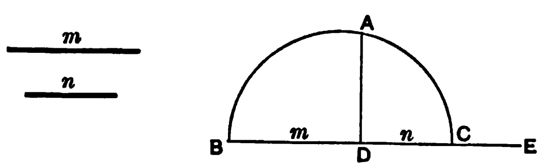


AD will be a mean proportional between BD and DC; AB will be a mean proportional between BC and BD; and AC will be a mean proportional between BC and DC.

302. *Exercise.*—Given the two perpendicular sides of a right triangle equal to 8 and 6 in. respectively to compute the length of the perpendicular from the vertex of the right angle to the hypotenuse.

303. *Exercise.*—If in a right triangle one of the sides about the right angle is double the other, what is the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse?

304. CONSTRUCTION. *To find a mean proportional between two given lines, m and n.*

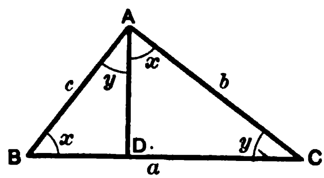


On the indefinite straight line BE lay off $BD = m$ and $DC = n$.
On BC as a diameter describe a semicircle.
At D erect DA perpendicular to BC, to meet the semicircle.
DA will be a mean proportional between m and n. § 301

Q. E. F.

PROPOSITION XV. THEOREM

305. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.**



GIVEN—the right triangle ABC right angled at A, with sides a, b, c.

TO PROVE $b^2+c^2=a^2$.

Draw AD perpendicular to the hypotenuse BC.

Then $\left.\begin{array}{l} b^2=a\times DC \\ c^2=a\times BD \end{array}\right\}$ § 298

Adding $b^2+c^2=a\times(BD+DC)=a\times a$. Ax. 2

Or $b^2+c^2=a^2$. Q. E. D.

306. COR. I. *The square of either side about the right angle is equal to the difference of the squares of the other two sides.*

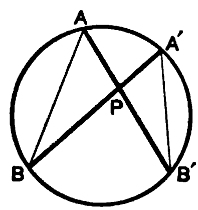
307. COR. II. *The diagonal of a square is equal to the side multiplied by the square root of two.*

Hint.—The square of the diagonal is equal to the sum of the squares of two sides.

* This proposition was first discovered by Pythagoras in the form given in Book IV., Proposition XI. But the Egyptians are supposed to have known as early as 2000 B.C. how to make a right angle by stretching around three pegs a cord measured off into 3, 4, and 5 units. The ancient Hindoos and Chinese also used this method. It is doubtful, however, whether the fact that $3^2+4^2=5^2$ was ever observed by them. It may be noted that essentially this method of forming a right angle is still used by carpenters. Sticks of 6 feet and 8 feet form two sides, and a "ten-foot pole" completes the triangle.

PROPOSITION XVI. THEOREM

308. *If through a fixed point within a circle two chords are drawn, the product of the two segments of one is equal to the product of the two segments of the other.*



GIVEN—P, a fixed point in a circle, and AB' and $A'B$ any two chords drawn through P.

TO PROVE $PA \times PB' = PB \times PA'$.

Join A and B, and A' and B'.

In triangles APB, $A'PB'$, the angles at P are equal. § 30

[Being vertical.]

Also the angles at A and A' are equal. § 189

[Being inscribed in the same segment.]

Hence the triangles are similar. § 263

Therefore PA, opposite $B : PA'$, opposite $B' :: PB$, opposite $A : PB'$, opposite A'. § 261

Whence $PA \times PB' = PB \times PA'$ § 237

Q. E. D.

PROPOSITION XVII. THEOREM

309. *If from a point without a circle a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.*

GIVEN—a fixed point P outside of a circle, PC a tangent, and PB a secant (Fig 1).

TO PROVE $\frac{PB}{PC} = \frac{PC}{PA}$.

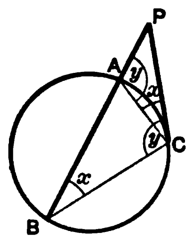


FIG. 1

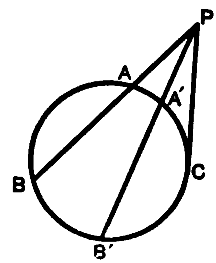


FIG. 2

Join A and C, and B and C. The triangles PAC and PCB have the angle at P common, and the angles PCA and PBC (both marked x) equal, each being measured by one-half the arc AC. §§ 189, 197

Therefore the triangles are similar. § 263

Hence PB, opposite y in large triangle : PC, opposite y in small : : PC, opposite x in large : PA, opposite x in small.

Q. E. D.

310. COR. Hence, in Fig. 2,

$$PB \times PA = \overline{PC}^2$$

and $$PB' \times PA' = \overline{PC}^2.$$

Therefore $$PB' \times PA' = PB \times PA.$$ Ax. 1

Hence, *if from a point without a circle two secants be drawn, the product of one secant and its external segment is equal to the product of the other and its external segment.*

311. *Def.*—The **projection** of a straight line AB, upon another straight line MN, is the portion of MN included between the perpendiculars let fall from the extremities of AB upon MN.

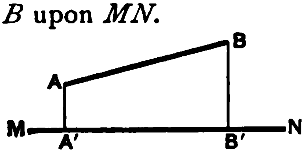


FIG. 1

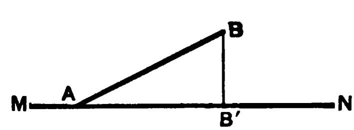


FIG. 2

In Fig. 1 $A'B'$ is the projection of AB. In Fig 2, where one extremity of AB is on MN, AB' is the projection.

PROPOSITION XVIII. THEOREM

312. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

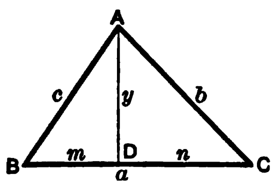


FIG. 1

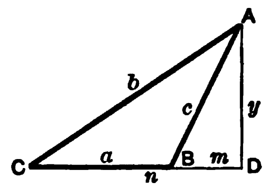


FIG. 2

GIVEN the triangle ABC and C, an acute angle.

Draw AD perpendicular to CB or CB produced, making CD the projection of AC on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $c^2=a^2+b^2-2an$.

In the right triangle ABD.

$$c^2=m^2+y^2. \quad (1)$$ § 305

In Fig. 1, $m=a-n$; and in Fig. 2, $m=n-a$.

In *both* cases $m^2=a^2-2an+n^2$.

Substituting this value in (1),

$$c^2=a^2-2an+n^2+y^2. \quad (2)$$

But in the triangle ACD, $n^2+y^2=b^2$. § 305

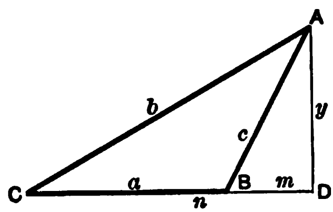
Substituting this value in (2),

$$c^2=a^2+b^2-2an.$$ Q. E. D.

SUMMARY: $c^2=m^2+y^2=a^2-2an+n^2+y^2=a^2-2an+b^2$.

PROPOSITION XIX. THEOREM

313. *In an obtuse-angled triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



GIVEN—the obtuse-angled triangle ABC with B the obtuse angle.

Draw AD perpendicular to CB produced, making BD the projection of AB on CB, and call $AB=c$; $AC=b$; $BC=a$; $AD=y$; $BD=m$; $CD=n$.

TO PROVE $b^2=a^2+c^2+2am$.

In the right triangle ACD

$$b^2=n^2+y^2. \quad (1)$$ § 305

But $n=a+m$.

And $n^2=a^2+2am+m^2$.

Substituting this value of n^2 in (1),

$$b^2=a^2+2am+m^2+y^2. \quad (2)$$

But in the triangle ABD, $m^2+y^2=c^2$. § 305

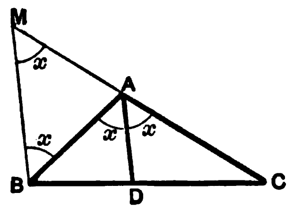
Substituting this value in (2),

$$b^2=a^2+c^2+2am.$$ Q. E. D.

SUMMARY: $b^2=n^2+y^2=a^2+2am+m^2+y^2=a^2+2am+c^2$.

PROPOSITION XX. THEOREM

314. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the other two sides.*



GIVEN—in the triangle ABC, AD the bisector of the angle A.

TO PROVE $$\frac{DC}{DB}=\frac{AC}{AB}.$$

Draw BM parallel to AD, meeting AC produced at M.

Then in the triangle BMC, since AD is parallel to BM,

$$\frac{DC}{DB}=\frac{AC}{AM}. \quad (1)$$ § 258

Also, since AD is parallel to MB,

angle $M=DAC$. § 48

[Being corresponding angles of parallel lines.]

And angle $MBA=BAD$. § 47

[Being alt.-int. angles of parallel lines.]

But angle $DAC=BAD$. Hyp.

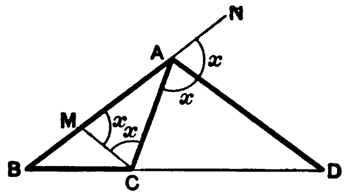
Therefore angle $M=MBA$. Ax. 1

And $AM=AB$. § 76

Substituting in (1), $$\frac{DC}{DB}=\frac{AC}{AB}.$$ Q. E. D.

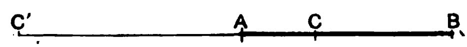
315. COR. Conversely, *if AD divides BC into two segments which are proportional to the adjacent sides, it bisects the angle BAC.*

316. *Exercise.*—The bisector of an exterior angle of a triangle meets the opposite side produced in a point whose distances from the extremities of that side are proportional to the other two sides.



Hint.—Prove $\frac{DB}{DC} = \frac{AB}{AM}$; prove also $AC = AM$, and substitute AC for AM in the proportion.

317. *Defs.*—The line AB is divided **internally** at C when this point is between the extremities of the line. The segments into which it is divided are CA and CB.



AB is divided **externally** at C' when this point is on the line produced. The segments are $C'A$ and $C'B$.

In each case the segments are the distances from the point of division to the extremities of the line. The line is the *sum* of the internal segments, and the *difference* of the external segments.

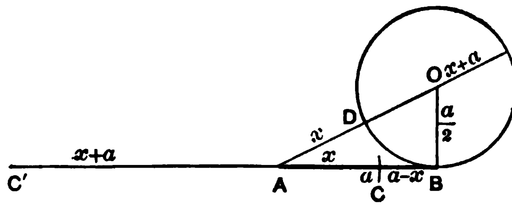
318. A line is divided **harmonically** when it is divided internally and externally in the same ratio.

Thus, if $\frac{CA}{CB} = \frac{C'A}{C'B}$, then AB is divided harmonically at C and C'.

319. *Exercise.*—Prove that the bisectors of the interior and exterior angles at one of the vertices of a triangle divide the opposite side harmonically.

320. *Def.*—A straight line is divided in **extreme and mean ratio** when one of its segments is a mean proportional between the whole line and the other segment.

321. CONSTRUCTION. *To divide a given straight line AB in extreme and mean ratio.*



At B draw the perpendicular $BO=\frac{AB}{2}=\frac{a}{2}$.

With the centre O and radius $\frac{a}{2}$ describe a circumference, and draw AO, cutting the circumference in D, D'.

On AB lay off $AC=AD$ $(=x)$, and extend BA to C', making $AC'=AD'$ $(=x+a)$; then also $BC'=x+2a$.

Then AB is divided in extreme and mean ratio, internally at C, and externally at C'.

Proof I. $AB^2=AD.\ AD'$, § 309

That is $a^2=x\ (x+a)$: (1)

Then $a\ (a-x)=x^2$,

Hence $\frac{a}{x}=\frac{x}{a-x}$,

That is $\frac{AB}{AC}=\frac{AC}{CB}$. Q. E. D.

II. Adding $ax+a^2$ to both sides of (1),

$$ax+2a^2=x^2+2\ ax+a^2:$$

Then $a\ (x+2a)=(x+a)^2$;

Hence $\frac{a}{x+a}=\frac{x+a}{x+2a}$,

That is $\frac{AB}{AC'}=\frac{AC'}{C'B}$. Q. E. D.

322. *Remark.*—AC and AC' may be computed in terms of a.

$$AO^2=AB^2+BO^2,$$

$$\left(x+\frac{a}{2}\right)^2=a^2+\frac{a^2}{4}=\frac{5}{4}a^2;$$

Hence $$x+\frac{a}{2}=a\frac{\sqrt{5}}{2}.$$

Therefore $$AC=x=a\left(\frac{\sqrt{5}-1}{2}\right).$$

And $$AC'=x+a=a\left(\frac{\sqrt{5}+1}{2}\right).$$

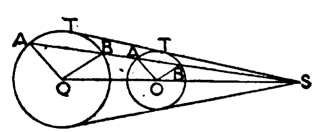
PROBLEMS OF DEMONSTRATION

323. *Exercise.*—The point of intersection of the internal tangents to two circles divides the line of centres internally into parts whose ratio is equal to the ratio of the radii.

324. *Exercise.*—The point of intersection of the external tangents to two circles divides the line of centres externally into parts whose ratio is equal to the ratio of the radii.

325. *Exercise.*—The points of intersection of the internal and external tangents to two circles divide the line of centres harmonically.

326. *Exercise.*—If through the centres of two circles two parallel radii are drawn in the same direction, the straight line joining their extremities will pass through the intersection of the external tangents.



327. *Exercise.*—If through the centres of two circles two parallel radii are drawn in opposite directions, the straight line joining their extremities will pass through the intersection of the internal tangents.

328. *Exercise.*—If through the intersection of the external or of the internal tangents to two circles a secant is drawn, the radii to the points of intersection will be parallel in pairs.

329. *Exercise.*—A triangle ABC is inscribed in a circle to which a second circle is externally tangent at A. If AB and AC are produced till they meet the second circumference at M and N, the triangles ABC and AMN are similar. §§ 197, 262

330. *Exercise.*—The perpendiculars from any two vertices of a

triangle on the opposite sides are inversely proportional to those sides. § 263

331. *Exercise.*—If two circles are tangent internally, all chords of the greater drawn from the point of contact are divided proportionally by the circumference of the smaller.

Hint.—Apply §§ 194, 211, 263.

332. *Exercise.*—On a common base AB are two triangles, ABC and ABC', whose vertices C and C' lie in a straight line parallel to AB. If a second parallel to AB cuts AC and BC in M and N, and AC' and BC' in M' and N', then $MN=M'N'$. § 262

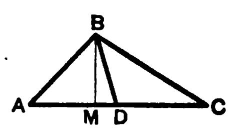
333. *Exercise.*—The difference of the squares of two sides of any triangle is equal to the difference of the squares of the projections of these sides on the third side. § 305

334. *Exercise.*—If from one of the acute angles of a right-angled triangle a straight line be drawn bisecting the opposite side, the square of that line will be less than the square of the hypotenuse by three times the square of half the side bisected.

335. *Exercise.*—If two circles intersect each other, the tangents drawn from any point of their common chord produced are equal. § 309

336. *Exercise.*—If two circles intersect each other, their common chord if produced will bisect their common tangent. § 309

337. *Exercise.*—I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, plus twice the square of the median drawn to the third side.

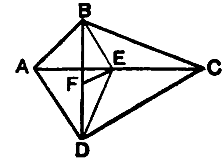


II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side and the projection of the median upon the third side.

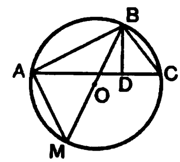
Hint.—The median BD divides ABC into two triangles, one acute angled and the other obtuse angled (provided AB and BC are not equal).

Apply §§ 312, 313.

338. *Exercise.*—In any quadrilateral the sum of the squares of the four sides is equal to the sum of the squares of the diagonals plus four times the square of the line joining the middle points of the diagonals.

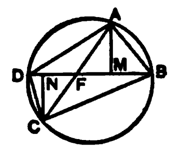


Hint.—Apply § 337, I. to the triangles ABC, ADC, and BED, and combine equations thus obtained.



339. *Exercise.*—The product of two sides of a triangle is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.

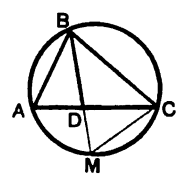
Hint.—Let ABC be the triangle. Draw the altitude BD and the diameter BM. Prove the triangles BAM and BDC similar. §§ 193, 194, 263



340. *Exercise.*—In an inscribed quadrilateral, $ABCD$, if F is the intersection of the diagonals AC and BD, then

$$\frac{AB \times AD}{CB \times CD} = \frac{AF}{FC}.$$

Hint.—In the triangles ABD and CBD, draw the altitudes AM and CN and apply § 339. Then compare triangles AFM and CFN.



341. *Exercise.*—The product of two sides of a triangle is equal to the square of the bisector of their included angle plus the product of the segments of the third side formed by the bisector.

Hint.—Circumscribe a circle about ABC and produce the bisector to cut the circumference in M. Prove the triangles ABD and MBC similar. Apply § 308.

PROBLEMS OF CONSTRUCTION

342. *Exercise.*—To produce a given straight line MN to a point X, such that $MN : MX = 3 : 7$.

343. *Exercise.*—To construct two straight lines having given their sum and ratio.

344. *Exercise.*—Having given the lesser segment of a straight line divided in extreme and mean ratio, to construct the whole line.

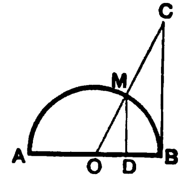
345. *Exercise.*—To construct a triangle having a given perimeter and similar to a given triangle.

346. *Exercise.*—To construct a right triangle having given an acute angle and the perimeter.

347. *Exercise.*—To divide one side of a given triangle into segments proportional to the other two sides.

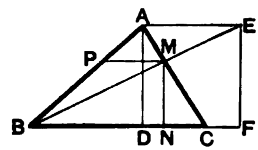
348. *Exercise.*—In a given circle to inscribe a triangle similar to a given triangle.

349. *Exercise.*—About a given circle to circumscribe a triangle similar to a given triangle.



350. *Exercise.*—To inscribe a square in a semicircle.

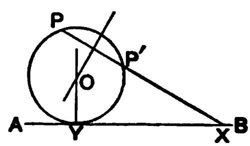
Hint.—At B draw CB equal and perpendicular to the diameter. Join OC cutting the circumference in M, and draw MD parallel to CB. Prove MD the side of the required square by § 262.



351. *Exercise.*—To inscribe a square in a given triangle.

Hint.—On the altitude AD construct the square $ADFE$ and draw BE cutting the side AC at M. From M draw MN and MP parallel to EF and AE respectively. Prove these lines equal and sides of the required square.

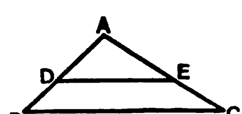
352. *Exercise.*—To inscribe in a given triangle a rectangle similar to a given rectangle.



353. *Exercise.*—To construct a circumference which shall pass through two given points and be tangent to a given straight line.

Hint.—Let AB be the given line, P and P' the points. If the straight line PP' is parallel to AB, the solution is simple. If PP' is not parallel to AB, it will cut it at some point X, and the distance from X to Y, the required point of tangency, may be determined by § 309.

PROBLEMS FOR COMPUTATION



354. (1.) In the triangle ABC, DE is drawn parallel to BC. If $\frac{AD}{DB}=\frac{4}{3}$, $BC=56$, and $AE=24$, find AC and DE.

(2.) The sides of a triangle are 3, 5, and 7. In a similar triangle the side homologous to 5 is equal to 65. Find the other two sides of the second triangle.

(3.) The shadow cast upon level ground by a certain church steeple is 27 yds. long, while at the same time that of a vertical rod 5 ft. high is 3 ft. long. Find the height of the steeple.

(4.) The footpaths on the opposite sides of a street are 30 ft. apart.

On one of them a bicycle rider is moving uniformly at the rate of 15 miles per hour. If a man on the other side, walking in the opposite direction, so regulates his pace that a tree 5 ft. from his path continually hides him from the rider, does he walk uniformly, and, if so, at what rate does he walk?

(5.) If from the top of a telegraph-pole standing upon the brink of a stream 23 m. wide a wire 30 m. long reaches to the opposite side of the stream, how high is the pole?

(6.) Given the two perpendicular sides of a right triangle equal to 25 cm. and 20 cm. respectively, to compute the length of the perpendicular from the vertex of the right angle to the hypotenuse.

(7.) If in a right triangle one of the sides about the right angle is three times the other, what is the ratio of the segments of the hypotenuse formed by the altitude on the hypotenuse?

(8.) There are two telegraph-poles standing upon the same level in a city street, one 59 ft. high, the other 45 ft. high, while between them, and in a straight line with their bases, is a hitching-post 3 ft. high. If the distance from the top of the post to the top of the higher pole is 100 ft., and from the top of the post to that of the lower pole 80 ft., how far apart are the poles?

(9.) If the chord of an arc is 720 ft. and the chord of its half is 369 ft., what is the diameter of the circle?

(10.) A chord of a circle is divided into two segments of 73.162 dcm. and 96.758 dcm. respectively by another chord, one of whose segments is 3.1527 m. What is the length of the second chord?

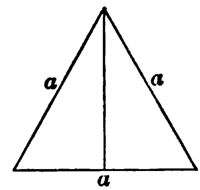
(11.) If from a point without a circle two secants are drawn whose external segments are 8 in. and 7 in., while the internal segment of the latter is 17 in., what is the length of the internal segment of the former?

(12.) In a triangle whose sides are respectively 25.136 cm., 31.298 cm., and 37.563 cm. in length, find the segments of the longest side formed by the bisector of the opposite angle.

(13.) If the base of an isosceles triangle is 60 cm., and each of its sides is 50 cm., find the length of its altitude in inches.

(14.) If the base of an isosceles triangle is b, and its altitude h, find the sides.

(15.) Find the altitude of an equilateral triangle whose side is 5 in.



(16.) If a is the side of an equilateral triangle, the altitude is $\frac{1}{2}a\sqrt{3}$.

(17.) Find in feet the side of an equilateral triangle having an altitude of 793.57 m.

(18.) In a right triangle, one of whose acute angles is 30°, and whose hypotenuse is a, the side opposite 30° is $\frac{1}{2}a$, and the other side is $\frac{1}{2}a\sqrt{3}$.

(19.) One acute angle of a right triangle is 30° and the hypotenuse is 4.3791 cm. Find the other sides.

(20.) Find the side of an isosceles right triangle whose hypotenuse is 3 ft.

(21.) If a is the hypotenuse of an isosceles right triangle, the side is $\frac{1}{2}a\sqrt{2}$.

(22.) Find the side of an isosceles right triangle whose hypotenuse is 32.174 dkm.

(23.) Find the base of an isosceles triangle whose side is 4 ft. and whose vertex angle is 30°.

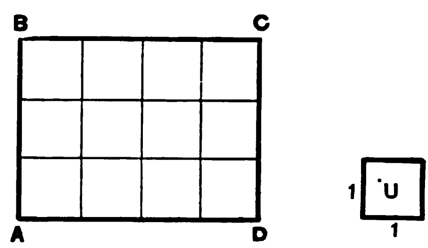
PLANE GEOMETRY

BOOK IV

AREAS OF POLYGONS

355. *Def.*—The **area** of a surface is the ratio of that surface to another surface taken as a unit.

The unit surface may have any size or shape, but the most common and convenient unit is a square having its side equal to the unit of length, as a square inch, a square mile, etc.



Thus, if U is the unit, the area of $ABCD$ is twelve.

356. *Def.*—**Equivalent** figures are figures having equal areas.

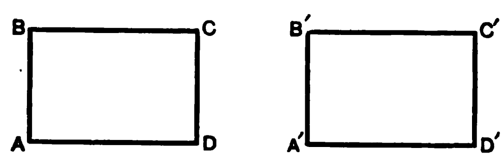
We may observe (1) figures of the same *shape* are *similar*.
(2) figures of the same *size* are *equivalent*.
(3) figures of the same *shape and size* are *equal*.

357. *Defs.*—The **bases** of a parallelogram are the side upon which it is supposed to stand and the opposite side.

The **altitude** is the perpendicular distance between the bases.

PROPOSITION I. THEOREM

358. *Two rectangles having equal bases and equal altitudes are equal.*



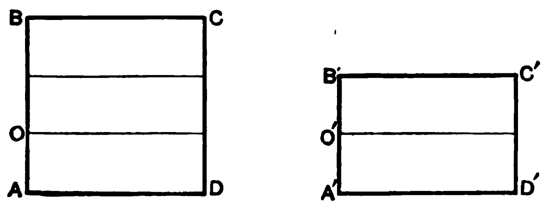
GIVEN—two rectangles, AC and $A'C'$, having equal bases, AD and $A'D'$, and equal altitudes, AB and $A'B'$.

TO PROVE the rectangles equal.

Make AD coincide with its equal $A'D'$.
Then AB will take the direction of $A'B'$. § 18
And B will fall on B'. Hyp.
That is, AB will coincide with $A'B'$.
Similarly DC will coincide with $D'C'$.
And therefore BC will coincide with $B'C'$. Ax. *a*
Hence the rectangles coincide throughout and are equal. § 15
Q. E. D.

PROPOSITION II. THEOREM

359. *The areas of two rectangles having equal bases are to each other as the altitudes.*



GIVEN—two rectangles AC and $A'C'$, having equal bases, AD and $A'D'$.

TO PROVE $$\frac{\text{rect. } AC^*}{\text{rect. } A'C'}=\frac{AB}{A'B'}.$$

CASE I. *When the altitudes, AB and $A'B'$, are commensurable.*

Suppose AO, the common measure of the altitudes, is contained in AB three times and in $A'B'$ twice.

Then $$\frac{AB}{A'B'}=\frac{3}{2}.$$ § 176

Through the several points of division draw parallels to the bases.

The rectangle AC will be divided into three rectangles and $A'C'$ into two, all of which will be *equal*. § 358

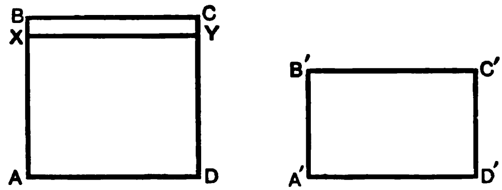
Hence $$\frac{\text{rect. } AC}{\text{rect. } A'C'}=\frac{3}{2}.$$ § 176

Therefore $$\frac{\text{rect. } AC}{\text{rect. } A'C'}=\frac{AB}{A'B'}.$$ Ax. 1

* A ratio of two rectangles or polygons means the ratio of their areas.

Thus, $\frac{\text{rect. } AC}{\text{rect. } A'C'}$ means $\frac{\text{the area of rect. } AC}{\text{the area of rect. } A'C'}$.

CASE II. *When the altitudes, AB and $A'B'$, are incommensurable.*



Suppose $A'B'$ to be divided into any number of equal parts and apply one of these parts to AB as a measure as often as it will be exactly contained.

Since AB and $A'B'$ are incommensurable, there will be a remainder XB, less than one of these parts.

Draw XY parallel to the base.

Since AX and $A'B'$ are constructed commensurable,

$$\frac{\text{rect. } AY}{\text{rect. } A'C'}=\frac{AX}{A'B'}. \qquad \text{Case I}$$

Now suppose the number of parts into which $A'B'$ is divided to be indefinitely increased.

We can thus make each part as small as we please.

But the remainder XB will always be less than one of these parts.

Therefore we can make XB less than any assigned quantity, though never zero.

That is, AX approaches AB as its limit. § 181

Likewise rect. AY approaches rect. AC as its limit.

Hence $\frac{AX}{A'B'}$ approaches $\frac{AB}{A'B'}$ as its limit.

Also $\frac{\text{rect. } AY}{\text{rect. } A'C'}$, approaches $\frac{\text{rect. } AC}{\text{rect. } A'C'}$, as its limit.

But since $$\frac{\text{rect. } AY}{\text{rect. } A'C'}=\frac{AX}{A'B'},$$

then $$\frac{\text{rect. } AC}{\text{rect. } A'C'}=\frac{AB}{A'B'}. \qquad \text{§ 182}$$

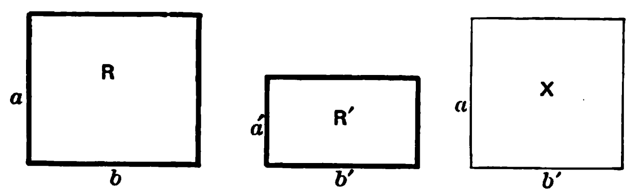
[If two variables are always equal and each approaches a limit, the limits are equal.] Q. E. D.

360. COR. *The areas of two rectangles having equal altitudes are to each other as their bases.*

Hint.—AD and $A'D'$ may be regarded as the altitudes, and AB and $A'B'$ as the bases.

PROPOSITION III. THEOREM

361. *The areas of any two rectangles are to each other as the products of their bases and altitudes.*



GIVEN—any two rectangles, R and R', their bases being b and b', and altitudes a and a'.

TO PROVE $\dfrac{R}{R'}=\dfrac{a\times b}{a'\times b'}$.

Construct rectangle X, having the same base as R' and altitude as R.

Then $\dfrac{R}{X}=\dfrac{b}{b'}$. § 360

[The areas of two rectangles having equal altitudes are to each other as their bases.]

And $\dfrac{X}{R'}=\dfrac{a}{a'}$. § 359

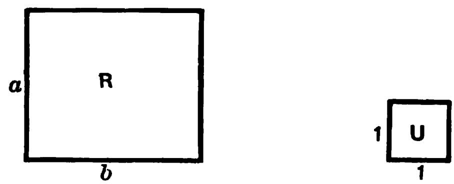
[The areas of two rectangles having equal bases are to each other as their altitudes.]

Multiplying, $\dfrac{R}{X}\times\dfrac{X}{R'}=\dfrac{b}{b'}\times\dfrac{a}{a'}$,

or $\dfrac{R}{R'}=\dfrac{a\times b}{a'\times b'}$. Q. E. D.

PROPOSITION IV. THEOREM

362. *The area of a rectangle is equal to the product of its base and altitude, provided the unit of area is a square whose side is the linear unit.*



GIVEN—the rectangle R and a square U with each side a linear unit.
TO PROVE—area of $R = a \times b$, provided U is the unit of area.

$$\frac{R}{U} = \frac{a \times b}{1 \times 1} = a \times b. \qquad \text{§ 361}$$

[The areas of two rectangles are to each other as the products of their bases and altitudes.]

But $$\frac{R}{U} = \text{area of } R. \qquad \text{§ 355}$$

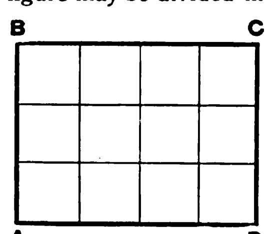
[The area of a surface is the ratio of that surface to the unit surface.]

Therefore . area of $R = a \times b$,
provided U is the unit of area. Ax. 1

Q. E. D.

363. *Remark.*—Hereafter it is to be understood without any express proviso that we take as the unit of area a square whose side is the linear unit.

364. *Remark.*—When the base and altitude of a rectangle each contain the linear unit an exact number of times, Proposition IV. becomes evident to the eye. Thus, if the base contain four and the altitude three linear units, the figure may be divided into twelve unit squares.

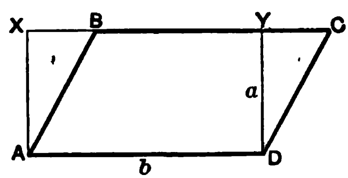


365. COR. *The area of a square is equal to the second power of its side.*

This fact is the origin of the custom of calling the second power of a number its "square."

PROPOSITION V. THEOREM

366. *The area of a parallelogram is equal to the product of its base and altitude.*



GIVEN—the parallelogram $ABCD$, with base b and altitude a.

TO PROVE the area of $ABCD = a \times b$.

Draw AX and DY perpendiculars between the parallels AD and BC.

Then $ADYX$ is a rectangle, having the same base and altitude as the parallelogram.

Right triangle AXB=right triangle DYC. (Why?)

Take away the right triangle DYC from the whole figure, and there is left the rectangle $ADYX$.

Take away the right triangle AXB from the whole figure, and there is left the parallelogram $ABCD$.

Therefore area $ADYX$=area $ABCD$. Ax. 3

But area $ADYX = a \times b$. § 362

[The area of a rectangle is equal to the product of its base and altitude.]

Therefore area $ABCD = a \times b$. Ax. 1

Q. E. D.

367. COR. I. *Parallelograms having equal bases and equal altitudes are equivalent.*

368. Cor. II.—*The areas of any two parallelograms are to each other as the products of their bases and altitudes.*

Hint.—Let the areas of the parallelograms be P and P', their bases b and b', and altitudes a and a'.

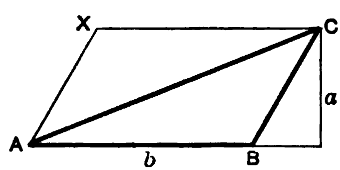
Then $P = ab$ and $P' = a'b'$.

And $\frac{P}{P'} = \frac{ab}{a'b'}$.

369. Cor. III.—*The areas of two parallelograms having equal bases are to each other as their altitudes: the areas of two parallelograms having equal altitudes are to each other as their bases.*

PROPOSITION VI. THEOREM

370. *The area of a triangle is equal to one-half the product of its base and altitude.*



Given the triangle ABC with base b and altitude a.

To prove area $ABC = \frac{1}{2} a \times b$.

From C draw CX parallel to AB.

From A draw AX parallel to BC.

Then the figure $ABCX$ is a parallelogram. § 112

and the triangle $ABC = \frac{1}{2}$ the parallelogram $ABCX$. § 114

[The diagonal of a parallelogram divides it into two equal triangles.]

But area paral. $ABCX = a \times b$. § 366

[The area of a parallelogram is equal to the product of its base and altitude.]

Therefore area triangle $ABC = \frac{1}{2} a \times b$. Ax. 8

Q. E. D.

371. Cor. I. *Triangles having equal bases and equal altitudes are equivalent.*

372. COR. II. *The areas of any two triangles are to each other as the products of their bases and altitudes.*

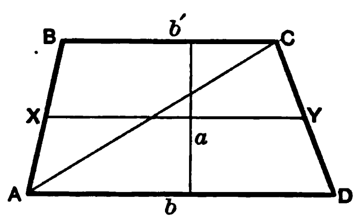
$$\left(\frac{P}{P'}=\frac{\frac{1}{2}\,ab}{\frac{1}{2}\,a'b'}=\frac{ab}{a'b'}.\right)$$

373. COR. III. *The areas of two triangles having equal bases are to each other as their altitudes: the areas of two triangles having equal altitudes are to each other as their bases.*

374. *Def.*—The **altitude** of a trapezoid is the perpendicular distance between its bases.

PROPOSITION VII. THEOREM

375. *The area of a trapezoid is equal to the product of its altitude and one-half the sum of its bases.**



GIVEN—the trapezoid $ABCD$ with altitude a and bases b and b'.

TO PROVE the area of $ABCD=\frac{1}{2}(b+b')\,a$.

Draw the diagonal AC.

Then area triangle $ADC=\frac{1}{2}\,ab,$ area triangle $ABC=\frac{1}{2}\,ab'.$ § 370

[The area of a triangle is equal to one-half the product of its base and altitude.]

Adding, area trapezoid $ABCD=\frac{1}{2}\,ab+\frac{1}{2}\,ab'.$ Ax. 11

$=\frac{1}{2}(b+b')\,a.$ Q. E. D.

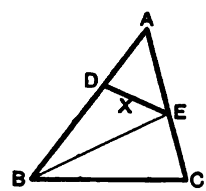
* The ancient Egyptians attempted to find the area of a field in the form of a trapezoid, in which $AB=CD$, by multiplying half the sum of its parallel sides by *one of its other sides*, an incorrect method.

376. COR. *The area of a trapezoid is equal to the product of its altitude and the line joining the middle points of the non-parallel sides.*

Hint.—Combine § 133 with the above proposition.

PROPOSITION VIII. THEOREM

377. *The areas of two triangles which have an angle of one equal to an angle of the other are to each other as the products of the sides including those angles.*



GIVEN—the triangles ADE and ABC placed so that their equal angles coincide at A.

TO PROVE $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$

Draw BE and denote the triangle ABE by X.

Then, regarding the bases of X and ADE as AB and AD, they will have a common altitude, the perpendicular from E to AB. Likewise X and ABC have bases AE and AC and a common altitude, the perpendicular from B to AC.

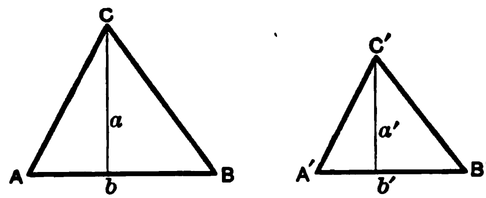
Therefore $$\left.\begin{aligned}\frac{\text{area } ADE}{\text{area } X}&=\frac{AD}{AB}\\ \text{and}\quad \frac{\text{area } X}{\text{area } ABC}&=\frac{AE}{AC}.\end{aligned}\right\}\qquad \S\ 373$$

[Triangles having equal altitudes are to each other as their bases.]

Multiplying, $$\frac{\text{area } ADE}{\text{area } ABC}=\frac{AD\times AE}{AB\times AC}.$$ Q. E. D.

PROPOSITION IX. THEOREM

378. *The areas of two similar triangles are to each other as the squares of any two homologous sides.*



GIVEN—two similar triangles ABC and $A'B'C'$, b and b' being homologous sides.

TO PROVE $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b^2}{b'^2}.$$

Draw the altitudes a and a'.

Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'}. \qquad \S\ 372$$

[The areas of two triangles are to each other as the products of their bases and altitudes.]

But $$\frac{a}{a'} = \frac{b}{b'}. \qquad \S\ 277$$

[Homologous altitudes of similar triangles have the same ratio as homologous sides.]

Substitute, in the previous equation, $\frac{b}{b'}$ for $\frac{a}{a'}$.

Then $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{b}{b'} = \frac{b}{b'} = \frac{b^2}{b'^2}.$$

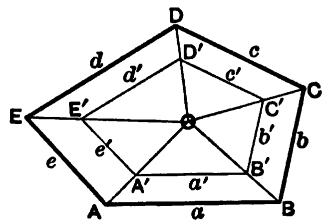
Q. E. D.

SUMMARY: $$\frac{\text{area } ABC}{\text{area } A'B'C'} = \frac{a \times b}{a' \times b'} = \frac{a}{a'} \times \frac{b}{b'} = \frac{b}{b'} \times \frac{b}{b'} = \frac{b^2}{b'^2}.$$

379. *Exercise.*—Prove the last proposition by means of Proposition VIII.

PROPOSITION X. THEOREM

380. *The areas of two similar polygons are to each other as the squares of any two homologous sides.*



GIVEN—the similar polygons $ABCDE$ and $A'B'C'D'E'$, with sides a, b, c, d, e, and a', b', c', d', e', and areas M and M' respectively.

TO PROVE $$\frac{M}{M'}=\frac{a^2}{a'^2}.$$

If $ABCDE$ and $A'B'C'D'E'$ are radially placed, the centre of similitude O being taken within the two polygons, the triangles OAB, OBC, OCD, etc., are respectively similar to $OA'B'$, $OB'C'$, $OC'D'$, etc. § 272

Then $$\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{a^2}{a'^2},\ \frac{\text{area } OBC}{\text{area } OB'C'}=\frac{b^2}{b'^2},\ \frac{\text{area } OCD}{\text{area } OC'D'}=\frac{c^2}{c'^2},\text{ etc.}$$ § 378

[The areas of two similar triangles are to each other as the squares of any two homologous sides.]

But $$\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2}=\text{etc.}$$ § 261

Hence $$\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{\text{area } OBC}{\text{area } OB'C'}=\frac{\text{area } OCD}{\text{area } OC'D'}=\text{etc.}=\frac{a^2}{a'^2}.$$ Ax. 1

Therefore $$\frac{\text{area } OAB+\text{area } OBC+\text{area } OCD+\text{etc.}}{\text{area } OA'B'+\text{area } OB'C'+\text{area } OC'D'+\text{etc.}}=\frac{a^2}{a'^2}.$$ § 251

But area OAB+area OBC+area OCD+etc.$=M$, Ax. 11

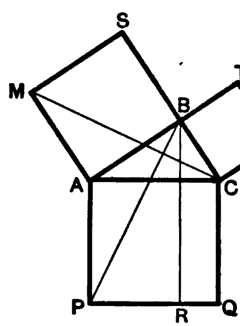
and area $OA'B'$+area $OB'C'$+area $OC'D'$+etc.$=M'$.

Therefore $$\frac{M}{M'}=\frac{a^2}{a'^2}.$$ Q. E. D.

381. COR. Since $\frac{a}{a'}$ = ratio of similitude, *the ratio of the areas of two similar polygons is equal to the square of their ratio of similitude.*

PROPOSITION XI. THEOREM

382. *The square described on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.**



GIVEN—the right triangle ABC and the squares described on its three sides.

TO PROVE—area of square AQ = area of square BN + area of square BM.

ABT and CBS are straight lines. § 29

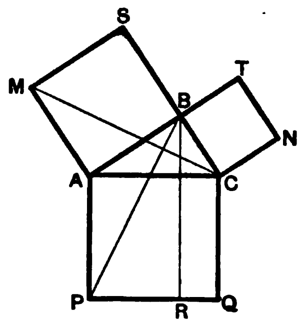
Join MC, and BP; draw BR parallel to AP.

Triangle† AMC = triangle ABP. § 78

[Having two sides and the included angle equal, viz., $AM = AB$, being

* Proposition XI. was discovered by Pythagoras (about 550 B.C.), and is usually known as the Pythagorean theorem. The proof here given is, however, due to Euclid (about 300 B.C.), that of Pythagoras being unknown.

† The eye will interpret this equality by conceiving the triangle AMC to turn around A as a pivot until AM falls on AB.



sides of a square; likewise $AC = AP$, and angle $CAM =$ angle PAB, since each consists of a right angle and the common angle BAC.]

But rectangle $AR \Bumpeq$ twice triangle ABP. §§ 362, 370

[Having the same base AP and the same altitude, the distance between the parallels AP and BR.]

Likewise square $BM \Bumpeq$ twice triangle AMC.

Therefore rectangle $AR \Bumpeq$ square BM. Ax. 7

Likewise it may be proved that

rectangle $CR \Bumpeq$ square BN.

Adding,

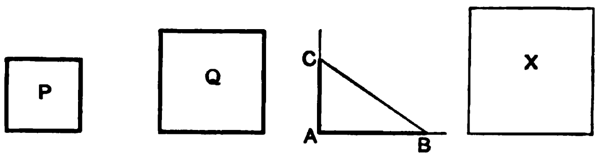
rect. $AR +$ rect. $CR \Bumpeq$ sq. $BM +$ sq. BN.

Or sq. $AQ \Bumpeq$ sq. $BM +$ sq. BN. Q. E. D.

383. COR. *The square on either side about the right angle is equivalent to the difference of the squares on the hypotenuse and on the other side.*

384. *Remark.*—Proposition XV., Book III., differs from the preceding proposition in that the squares of the sides in the former referred to the *algebraic* squares, that is, the second power of the numbers representing the sides, whereas in the latter case the squares are *geometric*. Inasmuch as the algebraic square measures the geometric square (§ 365), the truth of either of the two propositions involves the truth of the other.

385. CONSTRUCTION. *To construct a square equivalent to the sum of two given squares.*



GIVEN two squares P and Q.

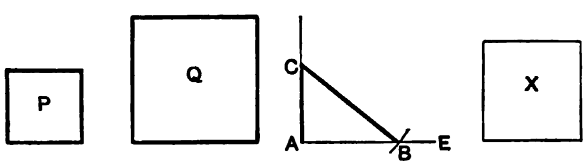
TO CONSTRUCT a square equivalent to $P+Q$.

Construct a right angle A and on its sides lay off AB and AC equal respectively to the sides of Q and P. Join B and C.

Construct the square X having its side equal to BC.

X is the required square. (Why?) Q. E. F.

386. CONSTRUCTION. *To construct a square equivalent to the difference of two given squares.*



GIVEN two squares, P and Q, of which P is the smaller.

TO CONSTRUCT a square equivalent to $Q-P$.

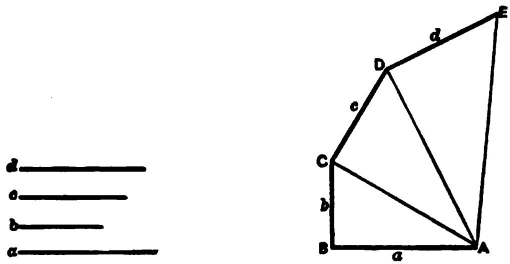
Construct a right angle A, and on one side lay off AC equal to the side of P.

Then from C as a centre, with the side of Q as a radius, describe an arc cutting AE at B.

Construct the square X having its side equal to AB.

X is the required square. (Why?) Q. E. F.

387. CONSTRUCTION. *To construct a square equivalent to the sum of any number of given squares.*



GIVEN a, b, c, d, the sides of given squares.

TO CONSTRUCT — a square equivalent to the sum of these given squares.

Draw AB equal to a.

At B draw BC perpendicular to AB and equal to b; join A and C.

At C draw CD perpendicular to AC and equal to c; join A and D.

At D draw DE perpendicular to AD and equal to d; join A and E.

The square constructed on AE as a side is the square required.

Proof.—Sq. on $AE \approx$ sq. on $d +$ sq. on AD.

$\approx$ sq. on $d +$ sq. on $c +$ sq. on AC.

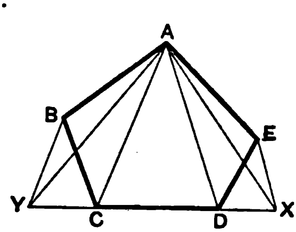
$\approx$ sq. on $d +$ sq. on $c +$ sq. on $b +$ sq. on a.

Q. E. F.

388. *Remark.*—The foregoing construction enables a draughtsman to construct a line whose length is equal to any square root.

Thus suppose it is necessary to construct a line equal to $\sqrt{3}$ inches. Lay off a, b, c, one inch each; then $AD = \sqrt{3}$ inches.

389. CONSTRUCTION. *To construct a triangle equivalent to a given polygon.*



GIVEN the polygon $ABCDE$.

TO CONSTRUCT a triangle equivalent to it.

Join any two alternate vertices as A and D.

Draw EX parallel to AD and meeting CD produced at X. Join A and X.

The polygon $ABCX$ has one less side than the original polygon, but is equivalent to it.

For the part $ABCD$ is common,

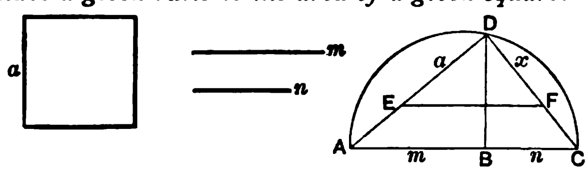
and triangle $ADE \approx$ triangle ADX. § 371

[Having the same base AD and the same altitude, the distance between the parallels AD and EX.]

In like manner reduce the number of sides of the new polygon $ABCX$, and continue until the required triangle AXY is obtained.

Q. E. F.

390. CONSTRUCTION. *To construct a square whose area shall have a given ratio to the area of a given square.*



GIVEN—a the side of a given square and $\frac{n}{m}$ the given ratio.

TO CONSTRUCT—a square which shall have the ratio $\frac{n}{m}$ to the given square.

Draw the straight line AB equal to m and produce it making BC equal to n.

Upon AC as a diameter construct a semicircle.

Erect the perpendicular BD meeting the circumference at D, and join DA and DC.

On DA lay off DE equal to a and draw EF parallel to AC.

Then DF, or x, is the side of the square required.

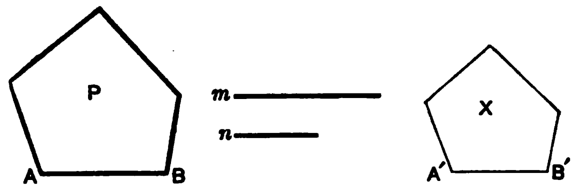
Proof: $\dfrac{\text{square on } x}{\text{square on } a}=\dfrac{x^2}{a^2}$ § 365

$$\left(\frac{x}{a}\right)^2=\left(\frac{DC}{DA}\right)^2=\frac{\overline{DC}^2}{\overline{DA}^2}=\frac{BC}{AB}.$$ §§ 259, 299

$$=\frac{n}{m}.$$

Q. E. D.

391. CONSTRUCTION. *To construct a polygon similar to a given polygon and whose area has a given ratio to that of the given polygon.*



GIVEN the polygon P, and the ratio $\frac{n}{m}$.

TO CONSTRUCT—a polygon similar to P, and which shall be to P as n is to m.

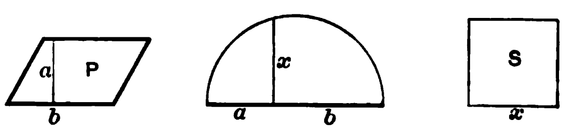
Find a line $A'B'$ such that the square upon it shall be to the square upon AB as n is to m. § 390

Upon $A'B'$, as the homologous side to AB, construct the required similar polygon X. § 290

Proof: $\frac{X}{P}=\frac{\overline{A'B'}^2}{\overline{AB}^2}=\frac{n}{m}$. (Why?)

Q. E. D.

392. CONSTRUCTION. *To construct a square equivalent to a given parallelogram.*



GIVEN a parallelogram P with base b and altitude a.

TO CONSTRUCT a square equivalent to P.

Construct x a mean proportional between a and b. § 304

Upon x construct the required square S.

Proof.—By construction $\frac{a}{x}=\frac{x}{b}$.

Hence $x^2=a\times b$. § 237

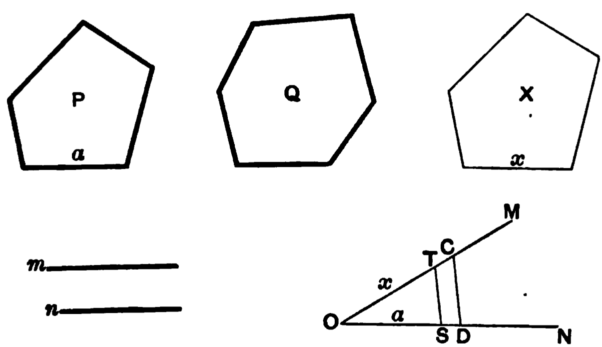
That is, area S = area P. §§ 365, 366

Q. E. D.

393. *Exercise.*—Show that a square can be constructed equivalent to a given triangle by taking for its side a mean proportional between the altitude and half the base.

394. *Exercise.*—Show that a square can be constructed equivalent to a given polygon by first reducing the polygon to an equivalent triangle and then constructing a square equivalent to the triangle.

395. CONSTRUCTION. ***To construct a polygon similar to a given polygon and equivalent to another given polygon.****



GIVEN the polygons P and Q.

TO CONSTRUCT—a polygon similar to P and equivalent to Q.

Construct squares equivalent to P and Q. § 394

Let n and m be the sides of these squares.

From any point O draw two lines OM and ON, and on these lay off OC equal to m and OD equal to n. On OD lay off OS equal to a, a side of P.

Draw parallels giving the fourth proportional OT. § 269

Upon OT, or x, as a side homologous to a, construct a polygon X similar to P. It will also be equivalent to Q.

Proof: $\frac{X}{P}=\frac{x^2}{a^2}=\frac{m^2}{n^2}=\frac{\text{sq. on } m}{\text{sq. on } n}=\frac{Q}{P}$. (Why?)

Therefore X is equivalent to Q and is similar to P by construction. Q. E. F.

* Pythagoras (about 550 B.C.) first solved this problem.

PROBLEMS OF DEMONSTRATION

396. *Exercise.*—The square on the base of an isosceles triangle, whose vertical angle is a right angle, is equivalent to four times the triangle.

397. *Exercise.*—A quadrilateral is divided into two equivalent triangles by one of its diagonals, if the second diagonal is bisected by the first.

398. *Exercise.*—The four triangles formed by drawing the diagonals of a parallelogram are all equivalent.

399. *Exercise.*—If from the middle point of one of the diagonals of a quadrilateral straight lines are drawn to the opposite vertices, these two lines divide the figure into two equivalent parts.

400. *Exercise.*—If the sides of any quadrilateral are bisected and the points of bisection successively joined, the included figure will be a parallelogram equal in area to half the original figure.

401. *Exercise.*—A trapezoid is divided into two equivalent parts by the straight line joining the middle points of its parallel sides.

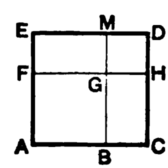
402. *Exercise.*—The triangle formed by joining the middle point of one of the non-parallel sides of a trapezoid to the extremities of the opposite side is equivalent to one-half the trapezoid.

403. *Exercise.*—If the three sides of a right triangle are the homologous sides of similar polygons described upon them, then the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the other two sides.

404. *Exercise.*—If M is the intersection of the medians of a triangle ABC, the triangle AMB is one-third of ABC.

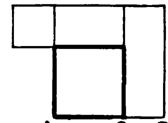
405. *Exercise.*—If from the middle point of the base of a triangle lines parallel to the sides are drawn, the parallelogram thus formed is equivalent to one-half the triangle.

406. *Exercise.*—The square described upon the sum of two straight lines is equivalent to the sum of the squares described upon the two lines plus twice their rectangle.



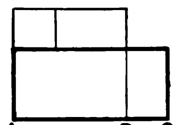
Hint.—Let AB and BC be the given lines.

407. *Exercise.*—Any straight line drawn through the intersection of the diagonals of a parallelogram divides the parallelogram into two equivalent parts.



408. *Exercise.*—The square described upon the difference of two straight lines is equivalent to the sum of the squares described upon the two lines minus twice their rectangle.

Hint.—Let AB and BC be the given lines.



409. *Exercise.*—The rectangle whose sides are the sum and the difference of two straight lines is equivalent to the difference of the squares described upon the two lines.

Hint.—Let AB and BC be the given lines.

Question.—To what three formulas of algebra* do the last three problems correspond?

PROBLEMS OF CONSTRUCTION

410. *Exercise.*—To divide a triangle into three equivalent triangles by straight lines from one of the vertices to the side opposite.

411. *Exercise.*—To construct an isosceles triangle equivalent to any given triangle, and having the same base.

412. *Exercise.*—On a given side, to construct a triangle equivalent to any given triangle.

413. *Exercise.*—Having given an angle and one of the including sides, to construct a triangle equivalent to a given triangle.

414. *Exercise.*—To construct a right triangle equivalent to a given triangle.

415. *Exercise.*—To construct a right triangle equivalent to a given triangle, and having its base equal to a given line.

416. *Exercise.*—On a given hypotenuse to construct a right triangle equivalent to a given triangle. When is the problem impossible?

417. *Exercise.*—To draw a straight line through the vertex of a given triangle so as to divide it into two parts whose areas have the ratio 2 to 5.

418. *Exercise.*—To bisect a triangle by a straight line drawn from a given point in one of its sides. § 377

* Euclid gave the geometric proofs of §§ 407–9; but though he may have translated them into algebra, he was probably not acquainted with the algebraic proof. To-day we find it easier to obtain the algebraic formulas first, and then give them the geometric interpretation. This is true in a multitude of cases where the opposite was true among the Greeks.

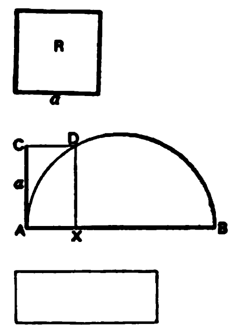
419. *Exercise.*—On a given side to construct a rectangle equivalent to a given square.

420. *Exercise.*—To construct a square equivalent to a given triangle.

421. *Exercise.*—To construct a square equivalent to the sum of two given triangles.

422. *Exercise.*—On a given side to construct a rectangle equivalent to the sum of two given squares.

423. *Exercise.*—To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.



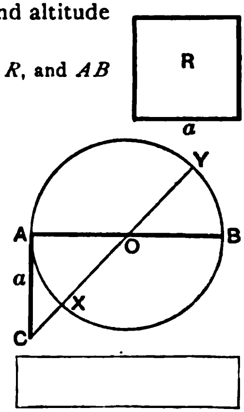
Hint.—Let a be the side of the given square R, AB the given line.

Upon AB as diameter construct a semicircle.

Draw CD parallel to AB at the distance a, intersecting the circumference in D, and draw DX perpendicular to AB.

The rectangle having its altitude equal to AX and its base equal to BX is the required rectangle.

424. *Exercise.*—To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.



Hint.—Let a be the side of the given square R, and AB the given line.

Upon AB as a diameter construct a circumference.

At A draw the tangent AC equal to a, and from C draw CXY through the centre.

Then the rectangle having its base equal to CY and its altitude equal to CX is the required rectangle.

425. *Exercise.* — To construct a square which shall have a given ratio to a given hexagon.

426. *Exercise.* — Through a given point within any parallelogram to draw a straight line dividing it into two equivalent parts.

PROBLEMS FOR COMPUTATION

427. (1.) Find the area of a parallelogram one of whose sides is 37.53 m., if the perpendicular distance between it and the opposite side is 2.95 dkm.

(2.) Required the area of a rhombus if its diagonals are in the ratio of 4 to 7, and their sum is 16.

(3.) In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the hypotenuse into the segments m and n. Find the area of the triangle.

(4.) If the hypotenuse of an isosceles right triangle is 30 ft., find the number of ares in its area.

(5.) Find the area of an isosceles right triangle if the hypotenuse is equal to a.

(6.) If one of the equal sides of an isosceles triangle is 17 dkm. in length and its base is 30 m., find the area of the triangle.

(7.) Find the area of an isosceles triangle if one of the equal sides is a and its base is b.

(8.) If in the above example $a=17.163$ hkm. and $b=27.395$ hkm., how many acres are there in the triangle?

(9.) Find the area of an equilateral triangle if one of the sides is 16 m.

(10.) If the side of an equilateral triangle is a, find its area.

(11.) If each side of a triangular park measures 196.37 rds., how many hectares does it contain?

(12.) If the perimeter of an equilateral triangle is 523.65 ft., find its area.

(13.) Find the area of a triangle, if two of its sides are 6 in. and 7 in. and the included angle is 30°.

(14.) Show that, if a and b are the sides of a triangle, the area is $\frac{1}{4}ab$, when the included angle is 30° or 150°; $\frac{1}{4}ab\sqrt{2}$, when the included angle is 45° or 135°; $\frac{1}{4}ab\sqrt{3}$, when the included angle is 60° or 120°.

(15.) Find the area of a triangle, if two of its sides are 43.746 mm. and 15.691 mm., and the included angle is 120°.

(16.) How many square feet are there in the entire surface of a house 50 ft. long, 40 ft. wide, 30 ft. high at the corners, and 40 ft. high at the ridge-pole?

(17.) If the bases of a trapezoid are respectively 97 m. and 133 m., and its area is 46 ares, find its altitude.

(18.) Find the area of a trapezoid of which the bases are 73 ft. and 57 ft., and each of the other sides is 17 ft.

(19.) Find the area of a trapezoid of which the bases are a and b and the other sides are each equal to d.

(20.) If in the triangle ABC a line MN is drawn parallel to the side AC so that the smaller triangle which it cuts off equals one-third of the whole triangle, find MN in terms of AC.

(21.) Through a triangular field a path runs from one corner to a point in the opposite side 204 yds. from one end, and 357 yds. from the other. What is the ratio of the two parts into which the field is divided?

(22.) If a square and a rhombus have equal perimeters, and the altitude of the rhombus is four-fifths its side, compare the areas of the two figures.

(23.) The altitude upon the hypotenuse of an isosceles right triangle is 3.1572 m. Find the side of an equivalent square.

(24.) If the areas of two triangles of equal altitude are 9 hectares and 324 ares respectively, what is the ratio of their bases?

(25.) A triangle and a rectangle are equivalent. (*a.*) If their bases are equal find the ratio of their altitudes. (*b.*) Compare their bases if their altitudes are equal.

(26.) Two homologous sides of two similar polygons are respectively 12 m. and 36 m. in length, and the area of the first is 180 sq. m. What is the area of the second?

(27.) Two similar fields together contain 579 hectares. What is the area of each if their homologous sides are in the ratio of 7 to 12?

(28.) In a triangle having its base equal to 24 in. and an area of 216 sq. in., a line is drawn parallel to the base through a point 6 in. from the opposite vertex. Find the area of the smaller triangle thus formed.

(29.) The altitude of a triangle is a and its base is b; the altitude, homologous to a, of another triangle, similar to the first, is c. Find the altitude, base, and area of a triangle similar to the given triangles and equivalent to their sum.

(30.) Construct a square equivalent to the sum of the squares whose sides are 20, 16, 9, and 5 cm.

(31.) If to the base b of a triangle the line d is added, how much must be taken from its altitude h that its area may remain unchanged?

PLANE GEOMETRY

BOOK V

REGULAR POLYGONS AND CIRCLES. SYMMETRY WITH RESPECT TO A POINT

428. *Defs.*—A figure turns **half-way round** a point, if a straight line of the figure passing through the point turns through 180°, i. e., half of 360°.

A figure turns **one-third way round** a point, if a straight line of the figure passing through the point turns through 120°, i. e., one-third of 360°.

In general, a figure turns **one-n^{th} way round** a point if a straight line of the figure passing through the point turns through one-n^{th} of 360°.

429. *Exercise.*—If a figure is turned half-way round on a point as a pivot, i. e., so that *one* straight line of the figure passing through that point turns through 180°, prove that *every other* straight line of the figure passing through that point turns through 180°.

430. *Exercise.*—In the same case, prove that every straight line not passing through the pivot makes after the rotation an angle of 180° with its original position.

431. *Exercise.*—If a figure turns one-third way round, prove that every straight line, whether passing through the pivot or not, makes after the rotation an angle of 120° with its original position.

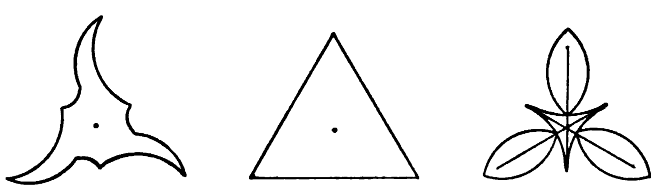
432. *Exercise.*—If a figure turns one-n^{th} way round, prove that every straight line of the figure makes after the rotation an angle equal to $\frac{1}{n}$ of $360°$ with its original position.

433. *Remark.*—Hence we see the propriety of saying that when one straight line of the figure turns through an angle, the whole figure turns through the same angle.

434. *Defs.*—A figure was defined to be symmetrical with respect to a point, called the **centre of symmetry** (§ 40), if, on being turned *half-way round* on that point as a pivot, the figure coincides with its original position or impression.

To distinguish this kind of symmetry from those which follow, it may be called **two-fold symmetry** with respect to a point.

435. *Def.*—A figure has **three-fold** symmetry with respect to a point, if, on being turned *one-third* way round on that point as a pivot, it coincides with its original impression.

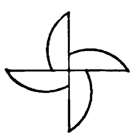


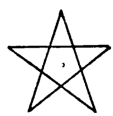
FIGURES POSSESSING THREE-FOLD SYMMETRY WITH RESPECT TO A POINT

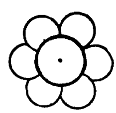
A figure which coincides with its original when turned one-third way round must also coincide when turned *two*-thirds. For, since it *coincides* after the first third, it may then be regarded as the original figure, and will therefore coincide when turned one-third again. When turned the third third the figure has completed one revolution, and each part is in its original position. It is easy to copy one of the above figures on tracing-paper or card-board, cut it out, fit it again to the page, stick a pin through its centre, and turn the figure one-third way round. In Propositions I. and II. it is convenient to think of the original diagram as fixed on the page, while another diagram, as the card-board, revolves upon it.

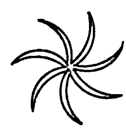
436. *Defs.*—Four-fold, five-fold, etc., symmetry may be defined likewise. In general a figure has **n-fold symmetry** with respect to a point, called the **centre of symmetry,** if, on being turned about that point one-n^{th} of a revolution, it coincides with its original impression.

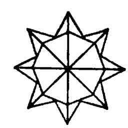
Such a figure will also coincide if turned an n^{th} of a revolution a second, third, fourth time, etc. For after the first n^{th} it becomes the *original figure,* and will therefore coincide when turned one-n^{th} again


4-FOLD SYMMETRY


5-FOLD SYMMETRY*


6-FOLD SYMMETRY

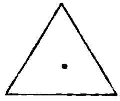

7-FOLD SYMMETRY

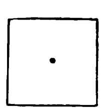

8-FOLD SYMMETRY

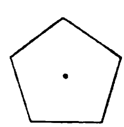
437. *Defs.*—A triangle is **regular,** if it has three-fold symmetry with respect to a point. The point is the **centre of the triangle.**

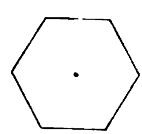
A quadrilateral is **regular,** if it has four-fold symmetry; a pentagon if it has five-fold symmetry, etc.

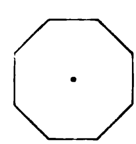
In general a polygon of n sides is **regular,** if it has n-fold symmetry. The centre of symmetry is the **centre of the polygon.**


REGULAR TRIANGLE


REGULAR QUADRILATERAL


REGULAR PENTAGON


REGULAR HEXAGON


REGULAR OCTAGON

* This figure was used as a badge by the secret society founded by Pythagoras about 550 B.C. for the pursuit of Mathematics and Philosophy. It was supposed to possess mysterious properties, and was called "Health."

PROPOSITION I. THEOREM

438. *Given a regular polygon :*

I. *All its sides are equal.*

II. *All its angles are equal.*

III. *A circle may be circumscribed about it, its centre being the centre of the polygon.*

IV. *A circle may be inscribed in it, its centre being the centre of the polygon.*

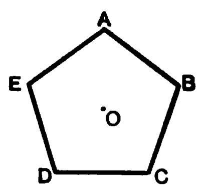


FIG. 1

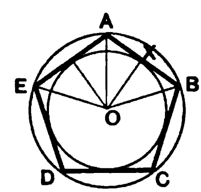


FIG. 2

GIVEN—*ABCDE*, a regular polygon of n sides with centre O.

TO PROVE—I. Its sides are equal.

II. Its angles are equal.

III. A circle can be circumscribed, with centre O.

IV. A circle can be inscribed, with centre O.

I. (Fig. 1.) By definition, the polygon will, after being turned about O one-n^{th} of a revolution, coincide with its original impression. § 437

Any side as AB must therefore take the position previously occupied by some other side.

Since each turn is one-n^{th} of a revolution, n turns are necessary before AB resumes its original position.

Hence in a complete revolution AB must coincide in succession with the n different sides of the polygon.

Hence AB is equal to each of the other sides, and they are all equal to each other. Q. E. D.

II. (Fig. 1.) Likewise any angle, as A, must in the n turns necessary for a complete revolution coincide in succession with the n different angles of the polygon.

Hence the angles are all equal. Q. E. D.

III. (Fig. 2.) Since the vertex A always remains at the same distance from O, it describes a circumference whose centre is O.

But it has been shown that the point A coincides successively with B, C, D, etc.

Hence the circumference described by A passes through B, C, D, etc.

That is, this circumference is circumscribed about the polygon and has for its centre the point O. § 205

Q. E. D.

IV. (Fig. 2.) Consider a perpendicular from O upon any side, as OX upon AB.

As the figure revolves, AB coincides successively with each of the other sides, and therefore OX becomes successively perpendicular to each side.

Hence the circumference generated by X, whose radius is OX, passes through the feet of all the perpendiculars from O to the sides.

The sides are therefore all tangent to this circle. § 170

That is, the circle is inscribed in the polygon, and has its centre at O. § 204

Q. E. D.

439. COR. I. *A regular triangle is an equilateral and equiangular triangle. A regular quadrilateral is a square.*

440. COR. II. *Each angle of a regular polygon is* $\frac{2n-4}{n}$ *right angles* (*n being the number of sides*).

Hint.—By § 65 the sum of all the angles is $2n-4$ right angles.

441. *Def.*—The **radius** of a regular polygon is the radius of the circumscribed circle, that is, the line from the centre to a vertex.

442. *Def.*—The **apothem** of a regular polygon is the radius of the inscribed circle, that is, the perpendicular from the centre to a side.

443. COR. III. *The angles at the centre of a regular polygon between successive radii are all equal, and each is one-n^{th} of four right angles.*

444. *Exercise.*—(1.) Find the number of degrees in an angle of each of the following regular polygons: (*a*) triangle, (*b*) pentagon, (*c*) hexagon, (*d*) octagon, (*e*) decagon, and (*f*) pentedecagon.

445. *Def.* Any one of these angles is usually spoken of simply as the **angle at the centre.**

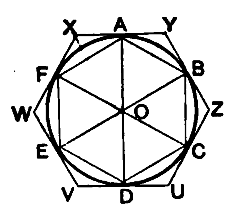
446. COR. IV. *The angle at the centre of a regular polygon is bisected by the apothem.*

PROPOSITION II. THEOREM

447. *If the circumference of a circle be subdivided into three or more equal arcs:*

I. *Their chords form a regular inscribed polygon, whose centre is the centre of the circle.*

II. *The tangents at the points of division form a regular circumscribed polygon, whose centre is the centre of the circle.*



GIVEN—a circle whose centre is O and whose circumference is divided into n equal arcs at the points A, B, C, D, etc.

TO PROVE—I. The n chords AB, BC, etc., form a regular polygon, with centre O.

II. The n tangents XAY, YBZ, etc., form a regular polygon, with centre O.

I. Revolve the figure one-n^{th} of 360°.

As the figure is turned, the circumference slides along itself. § 154

Since the arcs are each equal to one-n^{th} of the circumference, when A reaches B, B will reach C, C will reach D, etc.

That is, each vertex of the revolved polygon coincides with a vertex of the original polygon.

Since the vertices coincide, the sides which connect them must also coincide. Ax. *a*

Hence the whole polygon coincides with its original impression, and is therefore regular. § 437

Q. E. D.

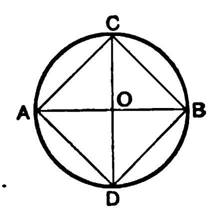
II. It has just been proved that if the figure is revolved one-n^{th} of 360°, the vertices A, B, C, etc., will coincide respectively with B, C, D, etc., and we know that the circumference will coincide with itself. § 154

Hence the tangents at A, B, C, etc., will coincide respectively with the tangents at B, C, D, etc. §§ 170, 18

Hence the whole circumscribed polygon will coincide with its original impression, and is therefore regular. § 437

Q. E. D.

448. CONSTRUCTION. *To inscribe a regular quadrilateral, or square, in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT an inscribed square.

Draw two perpendicular diameters AB and CD.

Join their extremities.

$ACBD$ is the required square.

Proof.—The arcs AC, CB, BD, DA are equal. § 158

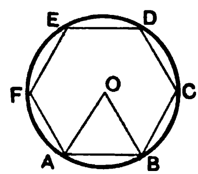
[Subtending equal angles at the centre.]

Hence $ABCD$ is a regular quadrilateral. § 447 I

Q. E. D.

449. *Remark.*—A regular polygon of eight sides can be inscribed by bisecting the arcs AC, CB, etc.; and, by continuing the process, regular polygons of sixteen, thirty-two, sixty-four, one hundred and twenty-eight, etc., sides can be inscribed.

450. CONSTRUCTION.—*To inscribe a regular hexagon in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed hexagon.

Draw any radius OA.

With A as a centre and a radius equal to OA describe an arc intersecting the circumference at B.

AB is a side of the required regular inscribed hexagon.

Proof.—Join O and B.

The triangle OAB is equilateral. Cons.

Hence angle O is 60°, i. e., one-sixth of 360°. § 73

Hence arc AB is one-sixth of the circumference. § 183

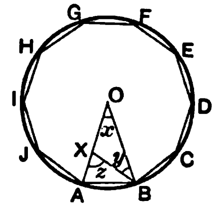
Therefore chord AB is a side of a regular inscribed hexagon. § 447 I

Q. E. D.

451. *Exercise.*—Show that a regular inscribed triangle is formed by joining the alternate vertices A, C, and E.

452. *Remark.*—A regular inscribed polygon of twelve sides can be formed by bisecting the arcs AB, BC, etc.; and, by continuing the process, regular polygons of twenty-four, forty-eight, ninety-six, etc., sides can be inscribed.

453. CONSTRUCTION. *To inscribe a regular decagon in a given circle.*



GIVEN a circle with centre O.

TO CONSTRUCT a regular inscribed decagon.

Divide a radius OA internally in extreme and mean ratio, i. e., so that

$$\frac{OA}{OX}=\frac{OX}{XA}. \qquad \S\ 320$$

With A as a centre and OX as a radius, describe an arc cutting the circumference at B.

AB is a side of the required regular inscribed decagon.

Proof.—Join BX and BO.

Substituting AB for its equal OX,

$$\frac{OA}{AB}=\frac{AB}{AX}.$$

Hence triangles AOB and ABX are similar. § 272

[Having the angle A common and the including sides proportional.]

But AOB is isosceles. § 146

Therefore ABX is isosceles, and $AB=BX=OX$. Cons.

Whence OXB is isosceles, and angle $y=$ angle x. § 70

Then angle $z=x+y=2x$. § 58

And angle $OBA=A=z=2x$. § 70

Hence, in the triangle AOB,

angle $OAB+OBA+x=5x=2$ right angles. § 57

Therefore $x=\frac{1}{5}$ of 2 right angles, or $\frac{1}{10}$ of 4 right angles.

And arc $AB=\frac{1}{10}$ of the circumference. § 183

Therefore chord $AB=$ side of regular inscribed decagon. § 447 I

Q. E. D.

454. *Exercise.*—Show that a regular pentagon can be inscribed by joining the alternate vertices, A, C, E, G, I.

455. *Remark.*—A regular polygon of twenty sides can be inscribed by bisecting the arcs AB, BC, etc., and by continuing the process regular polygons of forty, eighty, etc., sides can be inscribed.

456. *Exercise.*—To inscribe a regular pentedecagon in a given circle.

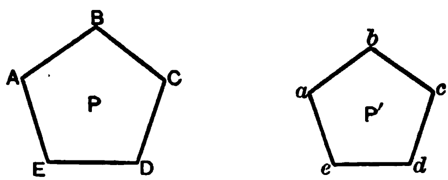
Hint.—Draw chords AB, the side of a regular inscribed hexagon, and AC, the side of a regular inscribed decagon.

Then chord BC is a side of a regular inscribed pentedecagon.

Discussion.—A regular polygon of thirty sides can be inscribed by bisecting the arcs CB, etc.; and by continuing the process regular polygons of sixty, one hundred and twenty, etc., sides can be inscribed.*

PROPOSITION III. THEOREM

457. *Two regular polygons of the same number of sides are similar.*



GIVEN—P and P', two regular polygons, each having n sides.

TO PROVE P and P' are similar.

$$\left.\begin{aligned} AB = BC = CD &= \text{etc.} \\ ab = bc = cd &= \text{etc.} \end{aligned}\right\} \qquad \S\, 438 \text{ I}$$

Dividing, $$\frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \text{etc.}$$

That is, the two polygons have their homologous sides proportional.

Also, since there are n angles in each polygon, each angle of either polygon contains $\frac{2n-4}{n}$ right angles. § 440

That is, the two polygons are mutually equiangular.

Therefore they are similar. § 261

Q. E. D.

* We have seen how to inscribe polygons of

3, 6, 12, 24, 48, 96, etc., sides,
4, 8, 16, 32, 64, 128, etc., sides,
5, 10, 20, 40, 80, 160, etc., sides,
15, 30, 60, 120, 240, 480, etc., sides.

Up to the year 1796 these were the only regular polygons for which constructions

PROPOSITION IV. THEOREM

458. *In two regular polygons of the same number of sides, two corresponding sides are to each other as the radii or as the apothems.*

GIVEN—AB and $A'B'$, sides of regular polygons, each having the same number (n) of sides; and OA, $O'A'$, and OF, $O'F'$, the radii and apothems respectively.

TO PROVE $$\frac{AB}{A'B'}=\frac{OA}{O'A'}=\frac{OF}{O'F'}.$$

In the triangles OAB and $O'A'B'$,

angle O = angle O'. § 443

[Each being one-n^{th} of four right angles.]

Also $OA = OB$ § 146

and $O'A' = O'B'$.

Whence $$\frac{OA}{O'A'}=\frac{OB}{O'B'}.$$

Therefore the triangles are similar. § 272

Hence $$\frac{AB}{A'B'}=\frac{OA}{O'A'}.$$ § 261

And $$\frac{AB}{A'B'}=\frac{OF}{O'F'}.$$ § 277

Q. E. D.

were known. In that year Gauss, the greatest mathematician of the nineteenth century, then nineteen years of age, discovered a method of constructing, by means of ruler and compasses, a regular polygon of 17 sides, and in general all polygons of $2^m(2^n+1)$ sides, m and n being integers, and (2^n+1) a prime number. This method was given in the *Disquisitiones Arithmeticæ*, published in 1801. In connection with this method Gauss enunciated the celebrated theorem that only a limited class of regular polygons are constructible by ruler and compass.

459. COR. I. *The perimeters of two regular polygons of the same number of sides are to each other as their radii or as their apothems.*

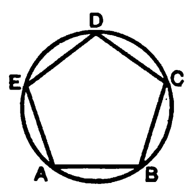
Hint.—Apply § 294.

460. COR. II. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii or as the squares of their apothems.*

461. *Exercise.* There are three regular hexagons; the side of the first is 20 in., that of the second is 1 m., that of the third 5 ft. Find in meters the side of a fourth regular hexagon whose area is equal to the sum of the areas of the first three.

PROPOSITION V. THEOREM

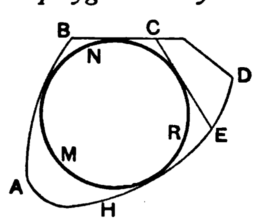
462. *The circumference of a circle is greater than the perimeter of an inscribed polygon.*



The proof is left to the student.

PROPOSITION VI. THEOREM

463. *The circumference of a circle is less than the perimeter of a circumscribed polygon or any enveloping line.*



GIVEN the circumference *MNR*.

TO PROVE—it is less than *ABCDEH*, any enveloping line.

Of all the lines enclosing the area MNR (of which the circumference MNR is one) there must be at least one ***shortest*** or ***minimum*** line.

The enveloping line $ABCDEH$ is not a minimum line, since a shorter one can be obtained by drawing a tangent CE.

For $CE < CDE$. § 7

Therefore $ABCEH < ABCDEH$. Ax. 4

Likewise it may be proved that ***every*** line enclosing MNR ***except*** the circumference is not minimum.

There remains therefore the circumference as the only minimum line. Q. E. D.

PROPOSITION VII. THEOREM

464. I. *If one regular inscribed polygon has twice as many sides as another, its perimeter and area are greater than those of the other.*

II. *If one regular circumscribed polygon has twice as many sides as another, its perimeter and area are less than those of the other.*

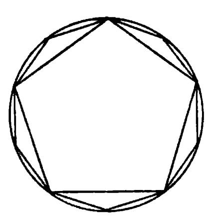


FIG. 1

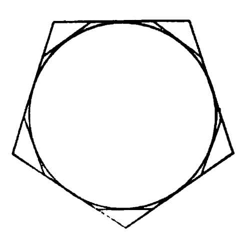


FIG. 2

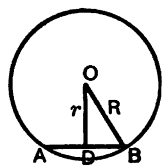
Hint.—If one polygon is entirely within another the first must be less than the second, unless the two coincide.

PROPOSITION VIII. LEMMA

465. *By doubling an indefinite number of times the number of sides of a regular polygon inscribed in a given circle:*

I. *The apothem can be made to differ from the radius by less than any assigned quantity.*

II. *The square of the apothem can be made to differ from the square of the radius by less than any assigned quantity.*



GIVEN—AB a side and r the apothem of a regular polygon inscribed in a circle whose radius is R.

TO PROVE—I. $R-r$ can be made as small as we please.

II. R^2-r^2 can be made as small as we please.

I. By doubling an indefinite number of times the number of divisions of the circumference, the arc AB can be made as small as we please.

Therefore the chord AB, which is always less than the arc, can be made as small as we please.

Therefore DB, half of that chord, can be made as small as we please.

But $R-r<DB$. § 134

Therefore $R-r$, which is always less than DB, can be made as small as we please. Q. E. D.

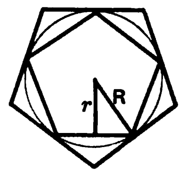
II. Since DB can be made as small as we please, $\overline{DB}^2$ can also be made as small as we please.

But $R^2-r^2=\overline{DB}^2$. § 306

Therefore R^2-r^2, the equal of $\overline{DB}^2$, can be made as small as we please. Q. E. D.

PROPOSITION IX. THEOREM

466. *The circumference of a circle is the limit which the perimeters of regular inscribed and circumscribed polygons approach when the number of their sides is doubled an indefinite number of times; and the area of the circle is the limit of the areas of these polygons.*



GIVEN—P and p the perimeters, R and r the apothems, S and s the areas, respectively, of regular circumscribed and inscribed polygons of the same number of sides.

TO PROVE—I. The circumference of the circle is the common limit of P and p, when the number of sides is doubled indefinitely.

II. The area of the circle is the common limit of S and s, when the number of sides is doubled indefinitely.

I. Since the two regular polygons have the same number of sides, $\frac{P}{p}=\frac{R}{r}$. § 459

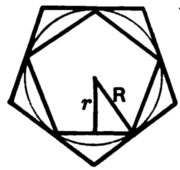
Hence $\frac{P-p}{P}=\frac{R-r}{R}$. § 247

Or $P-p=P\frac{R-r}{R}$.

But, by doubling indefinitely the number of sides, $R-r$ can be made as small as we please. § 465

Hence $\frac{R-r}{R}$, the preceding variable divided by R, a constant quantity, can be made as small as we please.

Hence $P\frac{R-r}{R}$, the preceding multiplied by P, a *decreasing* quantity (§ 464 II.), can be made as small as we please.



Hence its equal $P-p$ can be made as small as we please.

But the circumference is always less than P and greater than p. §§ 463, 462

Therefore P and p, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the circumference, by less than any assigned quantity.

But P and p can never equal the circumference. §§ 463, 462

Therefore by the definition of a limit the circumference is the common limit of P and p. § 181

Q. E. D.

II. Also, since the polygons are similar, § 457

$$\frac{S}{s}=\frac{R^2}{r^2}. \quad § 460$$

By division $$\frac{S-s}{S}=\frac{R^2-r^2}{R^2}.$$

Or $$S-s=S\frac{R^2-r^2}{R^2}.$$

But R^2-r^2 can be made as small as we please. § 465 II

Hence $\frac{R^2-r^2}{R^2}$, the preceding variable divided by R^2, a constant quantity, can be made as small as we please.

Hence $S\frac{R^2-r^2}{R^2}$, the preceding multiplied by S, a *decreasing* quantity (§ 464 II), can be made as small as we please.

Hence its equal $S-s$ can be made as small as we please.

But the area of the circle is always less than S and greater than s. Ax. 10

Therefore S and s, which can be made to differ *from each other* by less than any assigned quantity, can each be made to differ from the *intermediate quantity*, the area of the circle, by less than any assigned quantity.

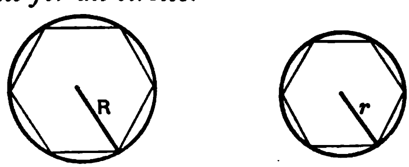
But S and s can never equal the area of the circle. Ax. 10

Therefore by the definition of a limit the area of the circle is the common limit of S and s. § 181

Q. E. D.

PROPOSITION X. THEOREM

467. *The ratio of the circumference of a circle to its diameter is the same for all circles.*



GIVEN—any two circles with radii R and r, and circumferences C and c respectively.

TO PROVE $$\frac{C}{2R}=\frac{c}{2r}.$$

Inscribe in the two circles regular polygons of the same number of sides, and call their perimeters P and p.

Then $$\frac{P}{p}=\frac{R}{r}=\frac{2R}{2r}.$$ § 459

Hence $$\frac{P}{2R}=\frac{p}{2r}.$$ § 243

As the number of sides of the two inscribed polygons is indefinitely doubled, P approaches C as its limit and p approaches c as its limit. § 466

Hence $\frac{P}{2R}$ approaches $\frac{C}{2R}$ as its limit,

and $\frac{p}{2r}$ approaches $\frac{c}{2r}$ as its limit.

But always $$\frac{P}{2R}=\frac{p}{2r}.$$

Hence $\frac{C}{2R}=\frac{c}{2r}$. § 182

Q. E. D.

468. *Def.*—This uniform ratio of a circumference to its diameter is called π. It will be shown in § 482 that its value is approximately $3\frac{1}{7}$.

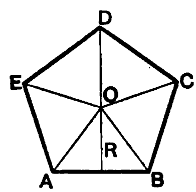
469. COR. *The circumference of a circle is equal to its radius multiplied by* 2π.

Hint.—By definition $\frac{C}{2R}=\pi$.

470. *Exercise.*—The radius of a locomotive driving-wheel is 6 feet; how far does the wheel roll on the track in one revolution?

PROPOSITION XI. THEOREM

471. *The area of a regular polygon is equal to one-half the product of its apothem and perimeter.*



GIVEN—a regular polygon $ABCDE$, R its apothem, and P its perimeter.

TO PROVE area polygon $=\frac{1}{2}R\times P$.

Draw from O the centre OA, OB, OC, etc.

The polygon is thus divided into as many triangles as it has sides.

The apothem R is their common altitude, and their bases are the sides of the polygon.

The area of *each* is $\frac{1}{2}$ R times its base. § 370

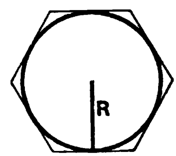
The area of *all* is $\frac{1}{2}$ R times the sum of their bases.

Or area polygon $=\frac{1}{2}R\times P$. Q. E. D.

472. Find the area of a regular hexagon circumscribed about a circle whose radius is R.

PROPOSITION XII. THEOREM

473. *The area of a circle is equal to one-half the product of its radius and circumference.*



GIVEN—a circle with radius R, circumference C, and area S.

TO PROVE $S=\frac{1}{2}R\times C$.

Circumscribe a regular polygon and call its perimeter C' and area S'.

Then $S'=\frac{1}{2}R\times C'$. § 471

[The area of a regular polygon is equal to one-half the product of its apothem and perimeter.]

Let the number of sides of the regular circumscribed polygon be indefinitely increased.

C', the perimeter of the polygon, approaches C, the circumference, as its limit. § 465

Hence $\frac{1}{2}R\times C'$ approaches $\frac{1}{2}R\times C$ as its limit.

Also S' approaches S as its limit. § 465

But always $S'=\frac{1}{2}R\times C'$.

Therefore $S=\frac{1}{2}R\times C$. § 182

Q. E. D.

474. COR. I. *The area of a circle is* πR^2.

475. COR. II. *The area of a sector whose angle is* $n°$ *is* $\frac{n}{360}(\pi R^2)$.

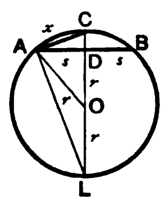
476. COR. III. *The areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

477. Find the ratio of the areas of two circles if the radius of one is the diameter of the other.

478. If the diameter of a given circle is 5 cm., find the diameter of a circle one-fourth as large.

PROPOSITION XIII. PROBLEM

479. *Given the radius of a circle and the side of a regular inscribed polygon, to find the side of a regular inscribed polygon of twice the number of sides.*



GIVEN—the circle O of radius r, and AB ($=2s$) the side of a regular inscribed polygon.

TO FIND—the length of AC ($=x$), the side of a regular inscribed polygon of twice the number of sides.

Draw CL the diameter perpendicular to AB.
Join AO and AL.

$AD=s$, § 164

Angle CAL is a right angle. § 194

Hence $\overline{AC}^2\ (=x^2)=CL\times CD$, § 298

$=2r\,(r-DO)$,

$=2r\,(r-\sqrt{\overline{AO}^2-\overline{AD}^2})$, § 306

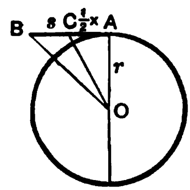
$=2r\,(r-\sqrt{r^2-s^2})$.

Therefore $x=\sqrt{2r\,(r-\sqrt{r^2-s^2})}$.

480. Prove that the side of a regular octagon, inscribed in a circle whose radius is r, is equal to $r\sqrt{2-\sqrt{2}}$.

PROPOSITION XIV. PROBLEM

481. *Given the radius of a circle and a side of a regular circumscribed polygon, to find the side of a regular circumscribed polygon of twice the number of sides.*



GIVEN—the circle O of radius r and AB $(=s)$, *half* the side of a regular circumscribed polygon.

TO FIND—2 AC $(=x)$ the side of a regular circumscribed polygon of twice the number of sides.

Join OA, OC, OB.

Angle AOB is *half* the angle between successive radii of the first polygon. § 446

Angle AOC is *half* the angle between successive radii of the second polygon. § 446

But the angle between successive radii in the second polygon is half that in the first. § 443

Therefore angle $AOC=\frac{1}{2}$ angle AOB, that is, OC bisects the angle AOB.

Hence $$\frac{AC}{CB}=\frac{AO}{OB},$$ § 314

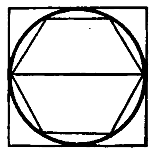
or $$\frac{AC}{AB-AC}=\frac{AO}{\sqrt{\overline{AO}^2+\overline{AB}^2}}.$$

Substituting, $$\frac{\frac{1}{2}x}{s-\frac{1}{2}x}=\frac{r}{\sqrt{r^2+s^2}}.$$

Solving, $$x=\frac{2rs}{r+\sqrt{r^2+s^2}}.$$

PROPOSITION XV. PROBLEM

482. *To compute the ratio of the circumference of a circle to its diameter approximately.*



GIVEN a circle.

TO FIND—the ratio of its circumference to its diameter approximately, or the value of π.

Since the ratio π is the same for all circles (§ 467), it is sufficient to compute it for any one.

Select a circle of which the diameter is unity.

The radius of this circle will be $\frac{1}{2}$; the side of a regular inscribed hexagon will be $\frac{1}{2}$, and of a circumscribed square 1.

Using the formula $x=\sqrt{2r(r-\sqrt{r^2-s^2})}$ (§ 479), which becomes when $r=\frac{1}{2}$, $x=\sqrt{\frac{1}{2}-\sqrt{\frac{1}{4}-s^2}}$, form the following table giving the length of the sides of regular inscribed polygons of 6, 12, 24, etc., sides. The length of the perimeter is obtained by multiplying the length of one side by the number of sides.

INSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
6	0.500000	3.000000
12	0 258819	3.105829
24	0.130526	3.132629
48	0.065403	3.139350
96	0.032719	3.141032
192	0.016362	3.141453
384	0.008181	3.141558

Using the formula $x = \frac{2rs}{r+\sqrt{r^2+s^2}}$ (§ 481), form the following table giving the length of the sides and perimeters of regular circumscribed polygons of 4, 8, 16, etc., sides.

CIRCUMSCRIBED REGULAR POLYGONS

NO. SIDES	LENGTH OF SIDE	LENGTH OF PERIMETER
4	1.000000	4.000000
8	0.414214	3.313709
16	0.198912	3.182598
32	0.098492	3.151725
64	0.049127	3.144118
128	0.024549	3.142224
256	0.012272	3.141750
512	0.006136	3.141632

But the length of the circumference must be between the lengths of the circumscribed and inscribed polygons. Hence it must be between 3.141558 and 3.141632. Hence 3.1416 is the nearest approximation to four decimal places.

Since the diameter of the circle is 1, the ratio of the circumference to the diameter is $\frac{3.1416}{1}$, or 3.1416.

That is, $\pi = 3.1416.$*

483. *Exercise.*—By means of the value of π just found and the formulas for the circumference and area of a circle, find the circumference and area of a circle whose radius is 23.16 inches.

* The earliest known attempt to obtain the area of the circle or to "square the circle" is recorded in a MS. in the British Museum recently deciphered. It was written by an Egyptian priest, *Ahmes*, at least as early as 1700 B.C., and possibly several centuries earlier. The method was to *deduct from the diameter of the circle one-ninth of itself and square the remainder.* This is equivalent to using a value of π equal to 3.16. *Archimedes* (about 250 B.C.), the greatest mathematician of ancient times, proved, by methods essentially the same as those employed in the text, that the true value of π lies between $3\frac{1}{7}$ and $3\frac{10}{71}$, i.e.,

PROBLEMS OF DEMONSTRATION

484. *Exercise.*—The angle at the centre of a regular polygon is the supplement of any angle of the polygon.

485. *Exercise.*—If the sides of a regular circumscribed polygon are tangent to the circle at the vertices of the similar inscribed polygon, then each vertex of the circumscribed figure lies in the prolongation of the apothem of the inscribed.

486. *Exercise.*—If the sides of a regular circumscribed polygon are tangent to the circle at the middle points of the arc, subtended by the sides of a similar inscribed polygon, then the sides of the circumscribed figure are parallel to those of the inscribed, and its vertices lie in the prolongation of the radii.

487. *Exercise.*—If from any point within a regular polygon of n sides perpendiculars are drawn to the several sides, the sum of these perpendiculars is equal to n times the apothem.

Hint.—Apply § 471.

488. *Exercise.*—The area of a circumscribed square is double that of an inscribed square.

between 3.1429 and 3.1408. *Ptolemy* (about 150 A.D.) used the value 3.1417. In the 16th century *Metrus*, of Holland, using polygons up to 1536 sides, obtained the easily remembered approximation $\frac{355}{113}$ (write 113355 and divide last three digits by first three), which is correct to six places of decimals. *Romanus*, also of Holland, using polygons of 1,073,741,324 sides, soon after computed sixteen places. With the better methods of higher mathematics various mathematicians have extended the computations gradually, until *Mr. Shanks*, in 1873, published a result to 707 places, the first 411 of which have been verified by *Dr. Rutherford.* The following are the first figures of his result:
π=3.141,592,653,589,793,238,462,643,383,279,502,884,197,169,399,375,105,8. How accurate a value this is may be inferred from Prof. Newcomb's remark that *ten* decimals would be sufficient to calculate the circumference of the earth to a fraction of an inch if we had an exact knowledge of the diameter.

The Greeks sought in vain for a perfectly accurate result or geometrical construction for obtaining a square equivalent to the circle, as did many mediæval mathematicians. "Circle squarers" still exist, although *Lambert* (about A.D. 1750) proved π incommensurable, *i. e.*, inexpressible as a finite fraction, and *Lindemann*, in 1882, proved it is also transcendental, *i. e.*, inexpressible as a radical or root of any algebraic equation with integral coefficients.

489. *Exercise.*—The side of an inscribed equilateral triangle is equal to one-half the side of a circumscribed equilateral triangle, and the area of the first is one-fourth that of the second.

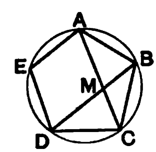
490. *Exercise.*—The apothem of an inscribed equilateral triangle is equal to half the radius.

491. *Exereise.*—The apothem of a regular inscribed hexagon is equal to half the side of the inscribed equilateral triangle.

492. *Exercise.*—The radius of a regular inscribed polygon is a mean proportional between its apothem and the radius of the similar regular circumscribed polygon.

493. *Exercise.*—The area of the ring included between two concentric circles is equal to that of a circle whose radius is one-half a chord of the outer circle drawn tangent to the inner.

494. *Exercise.*—In two circles of different radii, angles at the centre subtended by arcs of equal length are to each other inversely as their radii.



495. *Exercise.*—Two diagonals of a regular pentagon, not drawn from a common vertex, divide each other in extreme and mean ratio.

Hint.—Prove the triangles ABC and BCM similar (§ 262). Then prove $AM = AB = BC$ (§ 76), and substitute in the proportion derived from the first step.

PROBLEMS OF CONSTRUCTION

496. *Exercise.*—Having given a circle, to construct the circumscribed hexagon, octagon, and decagon.

497. *Exercise.*—Upon a given straight line as a side to construct a regular hexagon.

498. *Exercise.*—Having given a circle and its centre, to find two opposite points in the circumference by means of compasses only.

499. *Exercise.*—To divide a right angle into five equal parts.

500. *Exercise.*—To inscribe a square in a given quadrant.

501. *Exercise.*—Having given two circles, to construct a third circle equivalent to their difference.

502. *Exercise.*—To divide a circle into any number of equivalent parts by circumferences concentric with it.

PROBLEMS FOR COMPUTATION

503. (1.) If the side of a regular hexagon is 10 m., find the number of square feet in its area.

(2.) Find the area of a regular octagon whose radius is 12 cm.

(3.) If the radius of a circle is R, prove that the area of a regular inscribed dodecagon is $3R^2$.

(4.) If $R = 15.762$, find the length of a side and the apothem of a regular inscribed (*a*) triangle, (*b*) square, (*c*) hexagon.

(5.) A wheel, of radius 1.5 ft., made 3360 revolutions in going from one town to another. How many miles apart are the towns?

(6.) If the circumference of a circle is 50 in., find the radius.

(7.) If a wheel has 35 cogs, and the distance between the middle points of the cogs is 12 in., find the radius of the wheel.

(8.) Find the width of a ring of metal the outer circumference of which is 88 m. in length, and the inner circumference 66 m.

(9.) If the radius of a circle is 16 cm., how many degrees, minutes, and seconds are there in an arc 10 cm. long?

(10.) How many degrees are there in an arc whose length is equal to the radius of the circle?

(11.) If an arc of $30° = 12.5664$ in., find the radius of the circle.

(12.) Find the area of a circle whose radius is (*a*) 11 in.; (*b*) 17.146 m.

(13.) If the circumference of a circle is 60 ft., find the area.

(14.) The radius of a circle is 13 in. Find the side of a square whose area is equal to that of the circle.

(15.) What is the area of a circle inscribed in a square whose surface contains 211 ares?

(16.) Find the side of the largest square that can be cut from the cross-section of a tree 14 ft. in circumference.

(17.) A rectangle and a circle have equal perimeters. Find the difference in their areas if the radius of the circle is 9 in. and the width of the rectangle is three-fourths its length.

(18.) The chord of a segment of a circle is 34 in. in length, and the height of the segment is 8 in. Find the radius.

(19.) If the radius of a circle is 16 cm., what is the area of a sector having an angle of 24°?

(20.) The radius of a circle is 9 in. Find the area of a segment whose arc is 60°.

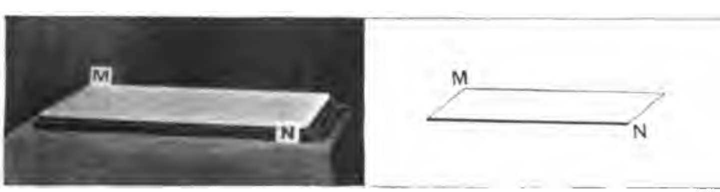
GEOMETRY OF SPACE

BOOK VI

STRAIGHT LINES AND PLANES

504. *Def.*—A **plane** has already been defined as "a surface such that, if any two points in it are taken, the straight line passing through them lies wholly in the surface." § 8

A plane is regarded as indefinite in extent, but is usually represented to the eye by a parallelogram lying in it.

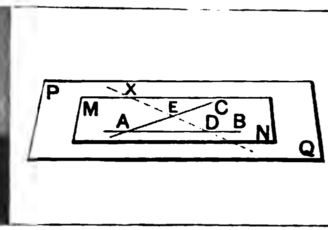


505. *Def.*—A plane is **determined** by given conditions, if it is the only plane fulfilling these conditions.

PROPOSITION I. THEOREM

506. *A plane is determined if it passes through:*

I. *Three points not in the same straight line.*
II. *A straight line and a point without that line.*
III. *Two intersecting straight lines.*
IV. *Two parallel straight lines.*



I. GIVEN—three points, A, B, and C not in the same straight line.

TO PROVE—that one and only one plane can be passed through them.

Pass a plane MN through one of the points, turn it about this point until it contains one of the other points, and then turn it about these two points until it contains the third.

No other plane will contain these points.

For, suppose PQ to be such a plane.

Take X *any point* in PQ. X will be proved to be also in MN.

Draw the straight lines AB and AC.

These will be in both planes, since A, B, and C lie in both planes. § 504

Through X draw a straight line in PQ cutting AB and AC in D and E.

Since D and E lie in the plane MN, the straight line DEX lies wholly in MN. § 504

Hence X, a point in DE, lies in the plane MN.

Thus ***any*** point, that is, ***every*** point in the plane PQ lies also in the plane MN, and in like manner it can be proved that every point in MN lies in PQ.

Therefore the two planes coincide. Q. E. D.

II. GIVEN—the straight line AB and the point C without AB.

TO PROVE—that one and only one plane can be passed through them.

The plane passed through C and any two points of AB will contain AB. § 504

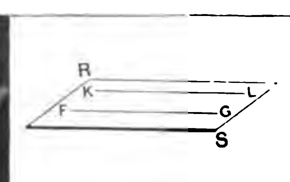
No other plane can be passed through AB and C, for then there would be two planes containing three points not in the same straight line, which is impossible. Q. E. D.

III. GIVEN—the straight lines AB and AC intersecting in A.

TO PROVE—that one and only one plane can be passed through them.

The plane passed through the three points A, B, and C will contain the straight lines AB and AC. § 504

No other plane can be passed through AB and AC, for then there would be two planes containing three points not in the straight line, which is impossible. Q. E. D.



IV. GIVEN—the parallel straight lines FG and KL.

TO PROVE—that one and only one plane can be passed through them.

These parallel lines lie in the same plane. § 31

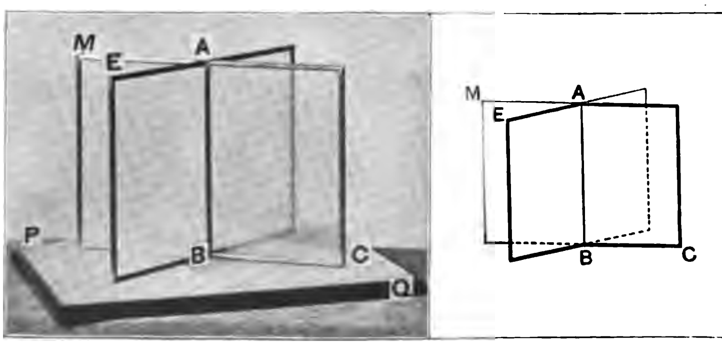
There cannot be two planes passed through them, for then there would be two planes containing three points, *F*, *G*, and *K*, not in the same straight line, which is impossible.

Q. E. D.

507. *Def.*—The **intersection** of two planes is the line common to both planes.

PROPOSITION II. THEOREM

508. *If two planes intersect, their intersection is a straight line.*



GIVEN two intersecting planes, *MB* and *EB*.

TO PROVE their intersection is a straight line.

If possible, suppose the intersection is not straight.

It would then contain three points not in the same straight line.

That is, the two planes would contain three points not in the same straight line, which is impossible. § 506 I

Therefore the intersection must be a straight line. Q. E. D.

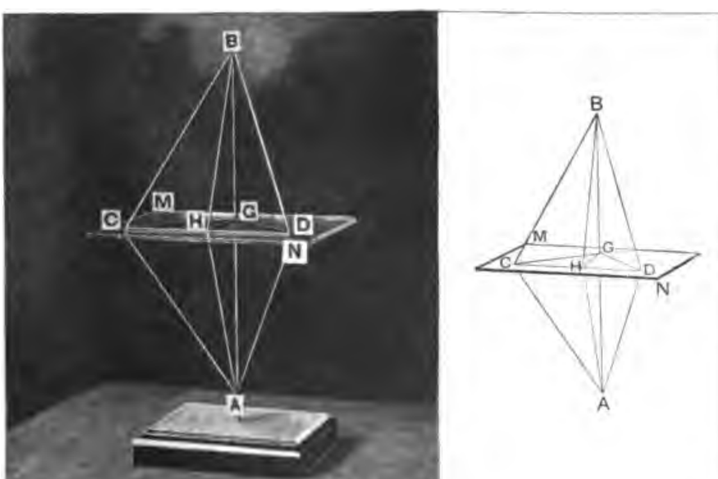
PERPENDICULAR AND OBLIQUE LINES AND PLANES

509. *Def.*—If a straight line meet a plane, its point of meeting is its **foot.**

510. *Defs.*—A straight line is **perpendicular** to a plane, if it is perpendicular to every straight line in the plane drawn through its foot. In the same case the plane is perpendicular to the line.

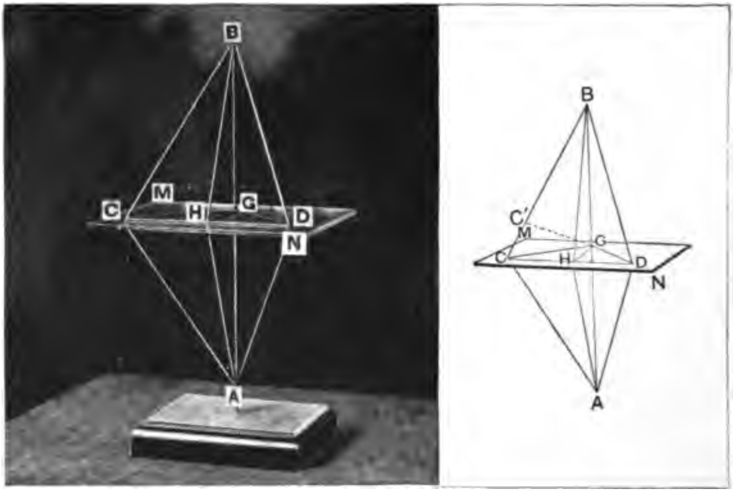
PROPOSITION III. THEOREM

511. *If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to the plane of those lines.*



GIVEN—the straight line BG perpendicular to each of the two intersecting straight lines GC and GD at the point G.

TO PROVE—that BG is perpendicular to the plane MN passed through GC and GD.



In the plane MN draw through G any straight line GH.

Let CD be any straight line cutting the lines GC, GH, and GD in C, H, and D.

Produce the line BG to A making GA equal to GB, and join A and B to C, H, and D.

Then, since GC is perpendicular to BA at its middle point,

$$CB = CA. \qquad \S\ 103$$

Similarly $$DB = DA.$$

Hence the triangles BCD and ACD are equal. § 88

And the homologous angles BCH and ACH are equal.

Hence the triangles BCH and ACH are equal. § 78

Therefore their homologous sides BH and AH are equal.

Therefore GH is perpendicular to BA. § 104

But GH is *any* straight line in MN passing through G.

Therefore *every* straight line in MN passing through G

is perpendicular to BA; that is, the plane MN is perpendicular to BA. § 510

Q. E. D.

512. COR. I. *At a given point in a straight line one and only one plane can be passed perpendicular to that straight line.*

Hint.—Let AB be the straight line and G the point.

At G draw the straight lines GC and GD perpendicular to AB.

The plane of these lines will be perpendicular to AB. (Why?)

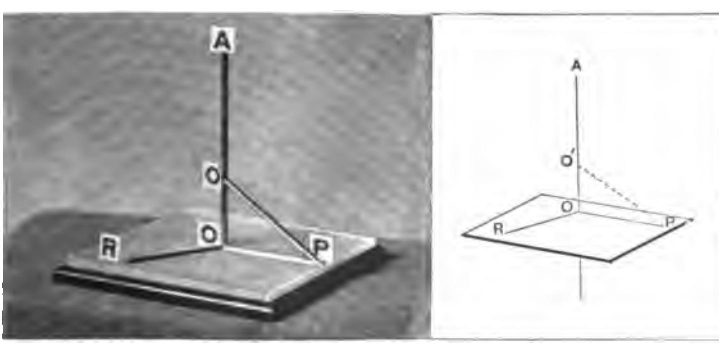
Only one such plane can be passed.

For any other plane passing through G cannot contain both of the lines GC and GD. (Why?)

It must therefore cut one of the planes BGC and BGD, as BGC, in some line GC' other than GC.

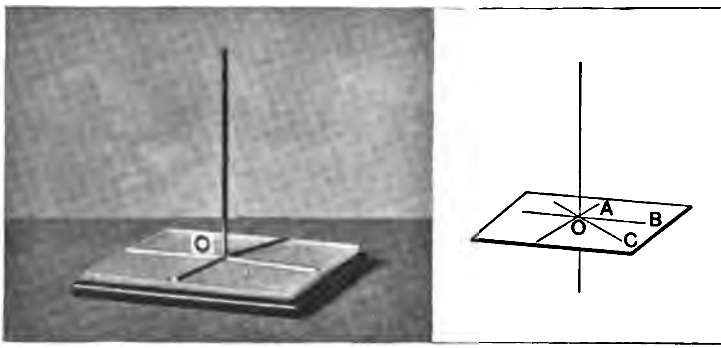
Since BGC' is not a right angle, this second plane is not perpendicular to AB. (Why?)

513. COR. II. *Through a given point without a straight line one and only one plane can be passed perpendicular to that straight line.*



Hint. — Use § 511 to pass *one* such plane. Any other plane cuts AO either at O or at some other point, O'. § 512 shows that the first is not perpendicular. Show also that the second is not.

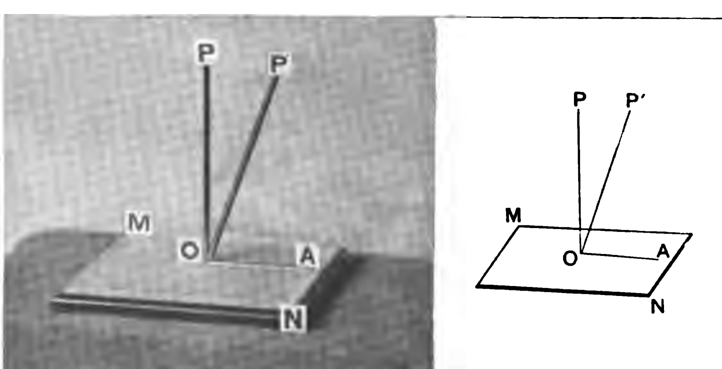
514. COR. III. *All the perpendiculars to a given straight line at the same point lie in a plane perpendicular to that line at that point.*



Hint.—Every pair of these perpendiculars, as OA and OB, determines a plane perpendicular at O. (Why?)

And all the planes thus determined must coincide. (Why?) Hence, etc.

515. COR. IV. *At a point in a plane one and only one perpendicular to the plane can be drawn.*

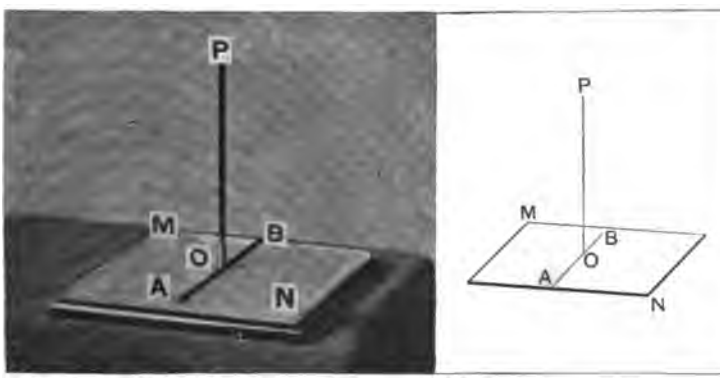


Hint.—Prove from Corollary I. that one perpendicular OP to the plane MN can be drawn.

No other line, as OP' through O can be perpendicular to MN.
For, let the plane of OP and OP' intersect MN in OA.
Since OP is perpendicular to OA, OP' is not. (Why?)
Therefore OP' is not perpendicular to MN. (Why?)

PROPOSITION IV. THEOREM

516. *The minimum line from a point to a plane is perpendicular to that plane.*



GIVEN—the plane MN, the point P without it, and PO, the minimum line from P to MN.

TO PROVE—that PO is perpendicular to MN.

In the plane MN through the point O draw *any* straight line AB.

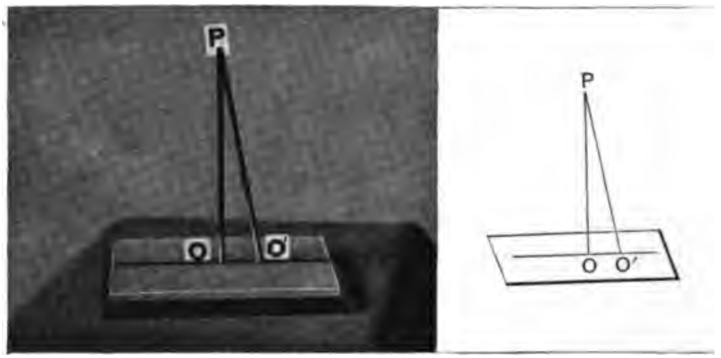
Since PO is the shortest line from P to the plane MN, it is the shortest line from P to the line AB in that plane.

Therefore PO is perpendicular to AB. § 95

That is, PO is perpendicular to *any* or *every* straight line in MN through its foot O.

Therefore PO is perpendicular to MN. Q. E. D.

517. COR. *From a point without a plane one and only one perpendicular to the plane can be drawn.*



Hint.—Apply the Proposition and § 34.

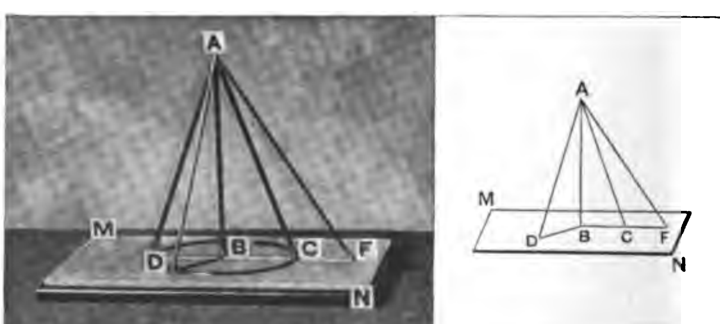
518. *Def.*—By the **distance from a point to a plane** is meant the shortest distance, and therefore the perpendicular distance.

PROPOSITION V. THEOREM

519. *If oblique lines are drawn from a point to a plane:*

I. *Those meeting the plane at equal distances from the foot of the perpendicular are equal.*

II. *Of two unequally distant, the more remote is the greater.*



I. GIVEN—the oblique lines AC and AD meeting the plane MN at the equal distances BC and BD from the perpendicular AB.

TO PROVE $AC=AD.$

In the triangles ABC and ABD, AB is common; $BC=BD$ by hypothesis; and the angles ABC and ABD are equal, being right angles.

Therefore the triangles are equal, and $AC=AD$. § 78

Q. E. D.

II. GIVEN—the oblique lines AF and AD meeting MN so that

$$BF>BD$$

TO PROVE $AF>AD$

On BF take $BC=BD$ and draw AC.

Then, from *plane* geometry, $AF>AC$. § 99

But $AD=AC$. CASE I

Therefore $AF>AD$. Q. E. D.

520. COR. Conversely:

I. *Equal oblique lines from a point to a plane meet the plane at equal distances from the foot of the perpendicular.*

II. *Of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

Hint.—Prove as in § 99.

521. *Remark.*—Article 520 supplies practical methods of drawing a straight line perpendicular to a plane, as a floor or a blackboard.

I. *To erect a perpendicular to a plane at a given point in it.*

With the given point as centre, describe a circumference in the given plane.

Take three strings of equal length somewhat longer than the radius of the circumference.

To each of three points on the circumference attach an end of one string.

Unite the three remaining ends in a knot and pull the strings taut.

A line through the given point and the knot is the perpendicular

required. Prove the method correct by supposing if possible that the foot of the perpendicular from the knot is not in the given point, and apply § 520 I.

II. *To draw a perpendicular to a given plane from a given point without it.*

From the point with a string of convenient length measure three equal distances to the plane.

The centre of the circumference which passes through the three points thus found is the foot of the required perpendicular. (Why?)

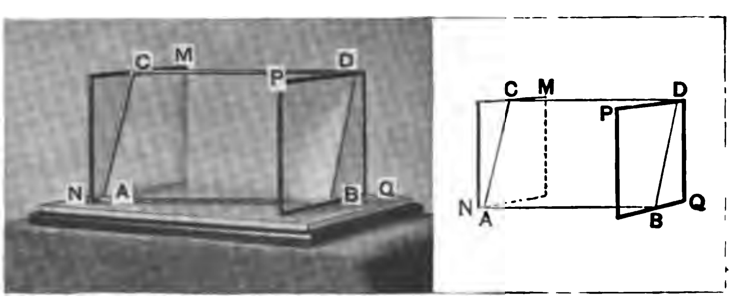
PARALLEL LINES AND PLANES

522. *Def.*—A straight line and a plane are **parallel** to each other if they cannot meet, however far produced.

523. *Def.*—Two planes are **parallel** to each other if they cannot meet, however far produced.

PROPOSITION VI. THEOREM

524. *If two parallel planes are cut by a third plane, their intersections with this plane are parallel.*



GIVEN—the parallel planes MN and PQ cut by the plane AD in the lines AC and BD.

TO PROVE AC and BD parallel.

Since the planes MN and PQ cannot meet, the lines AC and BD lying in them cannot meet.

Moreover these lines lie in the same plane AD.

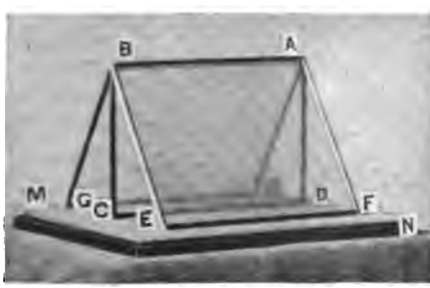
Therefore they are parallel. § 31

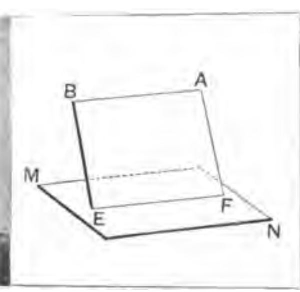
Q. E. D.

525. COR. *Parallel lines AB and CD intercepted between parallel planes are equal.*

PROPOSITION VII. THEOREM

526. *If a straight line is parallel to a plane, the intersection of the plane with a plane passed through the line is parallel to the line.*





GIVEN—the line BA parallel to the plane MN and a plane BF passing through BA and intersecting MN in EF.

TO PROVE BA parallel to EF.

These lines lie in the same plane.

They cannot meet, for BA cannot meet the plane MN in which EF lies.

Therefore they are parallel. § 31

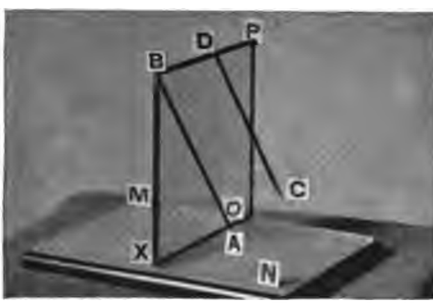
Q. E. D.

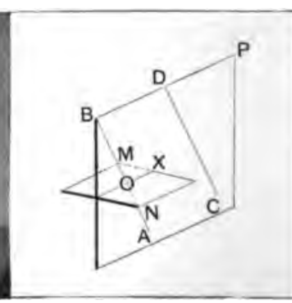
527. COR. *If two intersecting straight lines are parallel to a plane, their plane is parallel to the given plane.*

Hint.—If their plane were not parallel to the given plane it would intersect it in a line which would be parallel to both the given lines.

PROPOSITION VIII. THEOREM

528. *A plane which cuts one of two parallel lines must cut the other also.*





GIVEN—the parallel lines AB and CD, one of which, AB, is cut by the plane MN in the point O.

TO PROVE that CD is also cut by MN.

Pass a plane through AB and CD.

As this plane and the plane MN have the point O in common, their intersection must contain O. Call it OX.

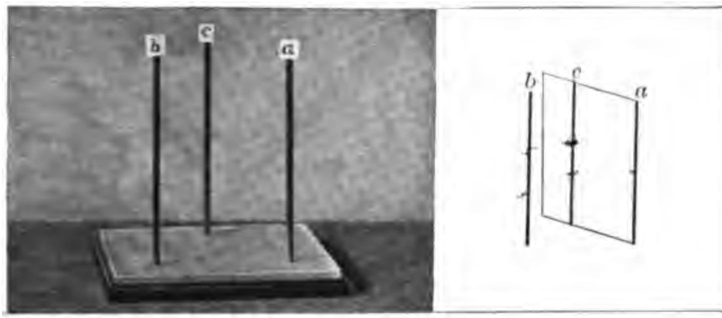
Suppose, if possible, that MN does not cut the line CD, but is parallel to it.

Then OX will also be parallel to CD. § 526

And there will be *two* lines, OX and OB through O, parallel to CD, which is impossible.

Therefore MN must cut CD. Q. E. D.

529. COR. I. *If two straight lines a and c are parallel to a third straight line b, they are parallel to each other.*

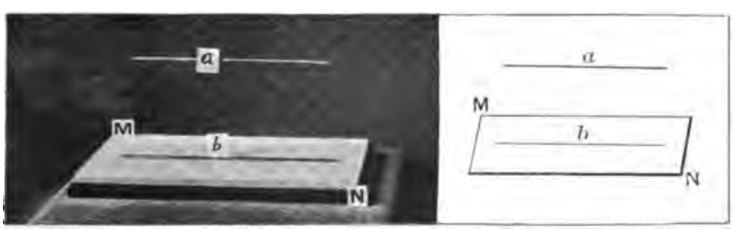


Hint.—Pass a plane through a and any point of c.

This plane will entirely contain c. Otherwise it would cut c and therefore b, which is parallel to c, and also a, which is parallel to b. This contradicts the hypothesis that it contains a.

Prove also that a and c cannot meet.

530. COR. II. *If two straight lines a and b are parallel, any plane MN, that contains one, as b, and not the other, is parallel to the second.*



Hint.—If MN is not parallel to a, it will cut it.

This is impossible, for then MN would cut b also.

Therefore MN is parallel to a.

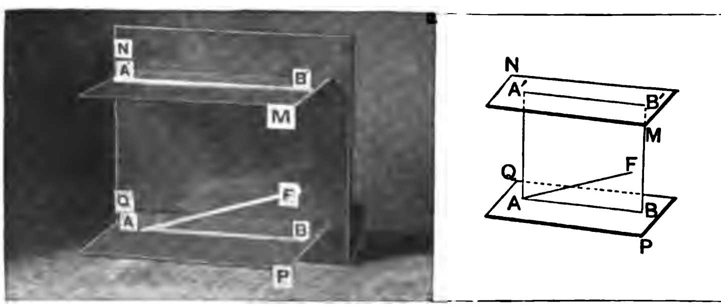
531. COR. III. *If two intersecting straight lines are parallel to two other intersecting straight lines, the plane of the first pair is parallel to the plane of the second pair.*

Hint.—Apply § 530 and then § 527.

PROPOSITION IX. THEOREM

532. *If two planes are parallel:*

I. *Any straight line which cuts one cuts the other.*
II. *Any plane which cuts one cuts the other.*



I. GIVEN—the parallel planes MN and PQ and the straight line AF cutting PQ in the point A.

TO PROVE—that AF is not parallel to MN but cuts MN.

Through AF and any point A' of MN not in AF pass a plane $A'B$.

Since this plane has a point in common with each of the parallel planes, it will intersect each in straight lines AB and $A'B'$.

These lines will be parallel. § 524

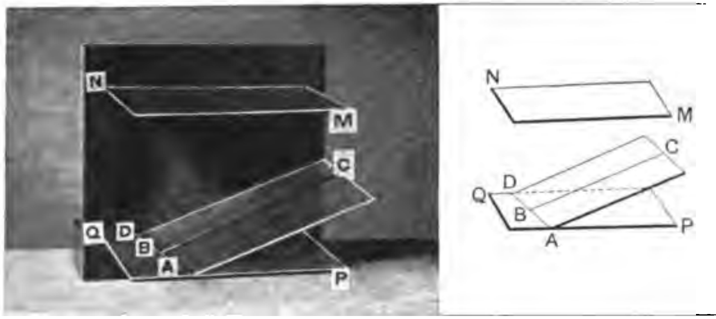
In the plane $A'B$ AF cuts AB, one of the two parallels AB and $A'B'$.

It therefore cuts the other, $A'B'$, since AF and AB cannot both be parallel to $A'B'$. Ax. *b*

Therefore AF cutting $A'B'$ cuts the plane MN in which $A'B'$ lies. Q. E. D.

II. GIVEN—the plane CD intersecting PQ in the straight line AD.

TO PROVE that CD also intersects MN.



In the plane CD draw any straight line BC cutting AD.

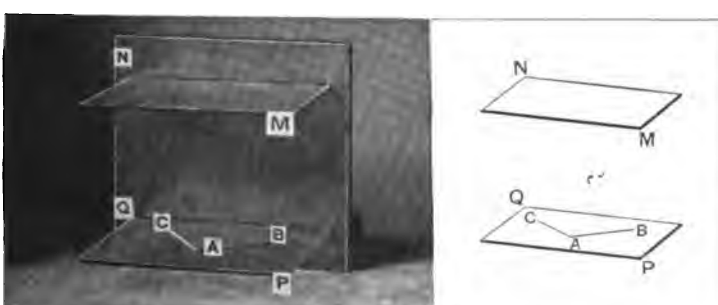
This line cuts PQ, and therefore cuts MN, by the first part of the proposition.

Therefore the plane CD, in which BC lies, cuts MN.

Q. E. D.

533. COR. I. *If two planes are parallel to a third plane they are parallel to each other.*

534. COR. II. *Through a given point without a given plane one plane can be passed parallel to the given plane, and but one.*



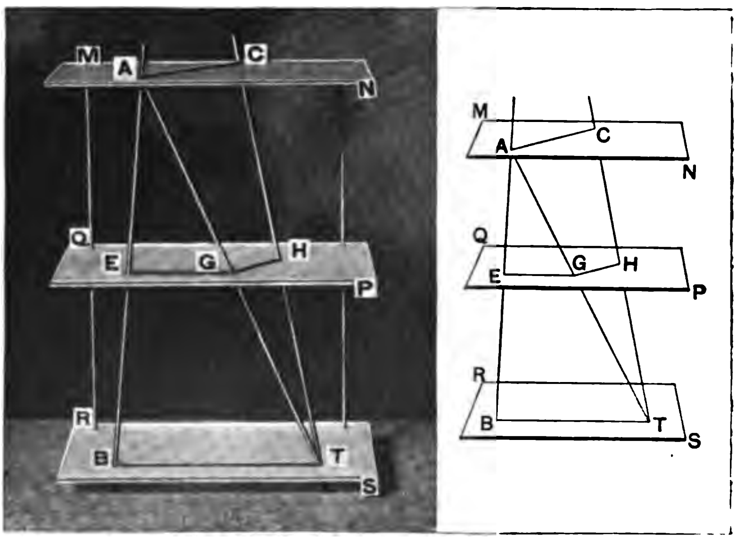
Hint.—Through the point A, without the plane MN, draw two straight lines AB and AC parallel to MN.

PQ, the plane of AB and AC, will be parallel to MN.

No other plane through A could be parallel to MN, for it would cut PQ, and therefore also MN.

PROPOSITION X. THEOREM

535. *If two straight lines are cut by three parallel planes, their corresponding segments are proportional.*



GIVEN—the straight lines AB and CT cut by the parallel planes MN, PQ, and RS in the points A, E, B, and C, H, T.

TO PROVE $\frac{AE}{EB}=\frac{CH}{HT}$.

Join A and T by a straight line cutting PQ in G.

EG and BT are the intersections of the plane of AB and AT with planes PQ and RS; also GH and AC are the intersections of the plane of TA and TC with PQ and MN. § 506 III

Then EG is parallel to BT and GH to AC. § 524

Therefore $\frac{AE}{EB}=\frac{AG}{GT}$, and $\frac{AG}{GT}=\frac{CH}{HT}$. § 258

Hence $\frac{AE}{EB}=\frac{CH}{HT}$. Ax. 1

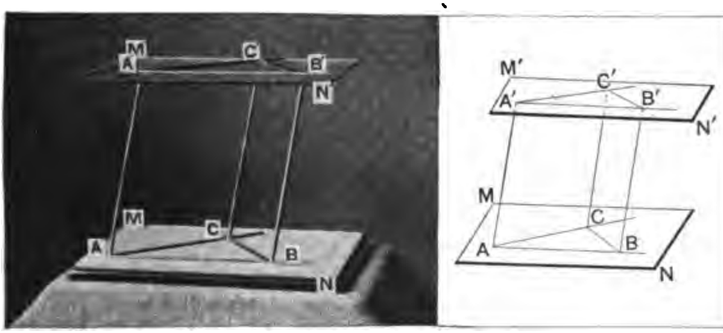
Q. E. D.

536. COR. *If a series of lines passing through a common point are cut by two parallel planes, their corresponding segments are proportional.*

Hint.—Pass a third plane through the common point parallel to one (and hence the other) of the two given planes.

PROPOSITION XI. THEOREM

537. *If two angles not in the same plane have their sides respectively parallel and extending in the same direction from their vertices, they are equal.*



GIVEN—the angles BAC and $B'A'C'$, whose sides, AB, $A'B'$, and AC, $A'C'$, are respectively parallel and extending in the same direction.

TO PROVE angle BAC = angle $B'A'C'$.

Take $AB=A'B'$ and $AC=A'C'$, and join AA', BB', CC'.

Then AB' and AC' will be parallelograms. § 123

Hence BB' and CC' are each equal and parallel to AA'. §§ 115, 112

Hence BB' and CC' are equal and parallel to each other. Ax. 1, § 529

Therefore BC' is a parallelogram, and $BC=B'C'$. § 123

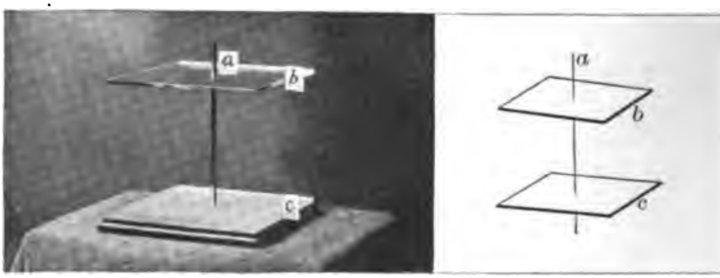
The triangles ABC and $A'B'C'$ are therefore equal. § 88

Hence angle BAC = angle $B'A'C'$. Q. E. D.

538. COR. *If two angles not in the same plane have their sides respectively parallel and extending in opposite directions from their vertices, they are equal; if two corresponding sides extend in the same direction, and the other two in opposite directions, the angles are supplementary.*

PROPOSITION XII. THEOREM

539. *If two planes are perpendicular to the same straight line, they are parallel.*



GIVEN—the planes b and c perpendicular to the straight line a.

TO PROVE b and c parallel.

If they should meet, there would be through any point of their intersection two planes, b and c, perpendicular to the same straight line a.

This is impossible. § 513

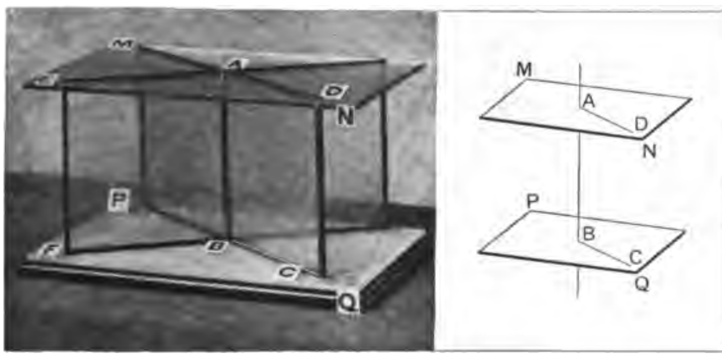
Therefore b and c are parallel. Q. E. D.

540. *Exercise.*—Prove this proposition as a consequence of §§ 33, 531.

Hint.—Pass two planes through a intersecting b and c in straight lines perpendicular to a.

PROPOSITION XIII. THEOREM

541. *If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other.*



GIVEN—the parallel planes MN and PQ, and the line AB perpendicular to MN at A.

TO PROVE AB perpendicular to PQ.

Since AB cuts MN, it also cuts PQ in some point B. § 532 I

[If two planes are parallel, any line that cuts one cuts the other.]

Through B draw in PQ any straight line BC, and through AB and BC pass a plane intersecting MN in AD.

Then BC is parallel to AD. § 524

[If two parallel planes are cut by a third plane, their intersections with this plane are parallel.]

But AB is perpendicular to AD. § 510

[A straight line perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.]

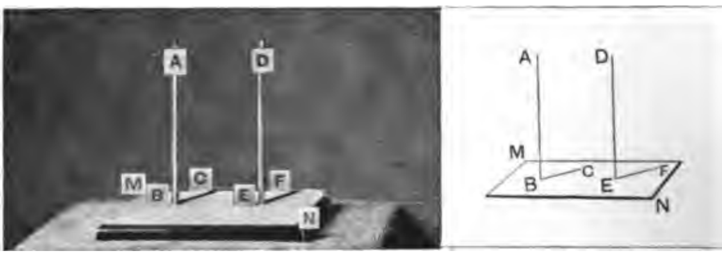
Therefore AB is also perpendicular to BC. § 36

Since AB is perpendicular to *any* straight line drawn in PQ through B, it is perpendicular to PQ. § 510

Q. E. D.

PROPOSITION XIV. THEOREM

542. *If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.*



GIVEN—the parallel lines AB and DE and the plane MN perpendicular to AB at B.

TO PROVE MN perpendicular to DE.

Since MN cuts AB, it also cuts DE in some point E. § 528

[If two lines are parallel, any plane that cuts one cuts the other.]

Through E draw in MN any straight line EF, and through B draw in MN the line BC parallel to EF.

Then angle DEF=angle ABC. § 537

But, since BC lies in MN, ABC is a right angle. § 510

Hence DEF is a right angle.

Since *any* straight line in MN through E is perpendicular to DE, MN is perpendicular to DE. Q. E. D.

543. COR. I. *If two straight lines are perpendicular to the same plane, they are parallel.*

Hint.—Suppose AB and DE perpendicular to MN.

Through any point of DE draw a line, as DE', parallel to AB.

Prove that DE and DE' coincide.

544. *Exercise.*—Prove § 529 by means of §§ 542, 543.

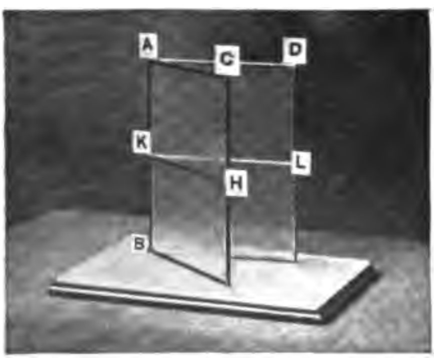
545. COR. II. *The perpendicular distance between two parallel planes is everywhere the same.*

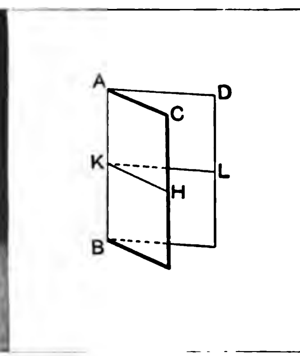
DIEDRAL ANGLES AND PROJECTIONS

546. *Defs.*—When two planes meet and are terminated at their common intersection, they form a **diedral angle.**

The planes are the **faces** of the diedral angle, and their intersection, the **edge.**

The faces are regarded as indefinite in extent.





A diedral angle may be designated by two points on its edge and one other point in each face, the former two being written between the latter two.

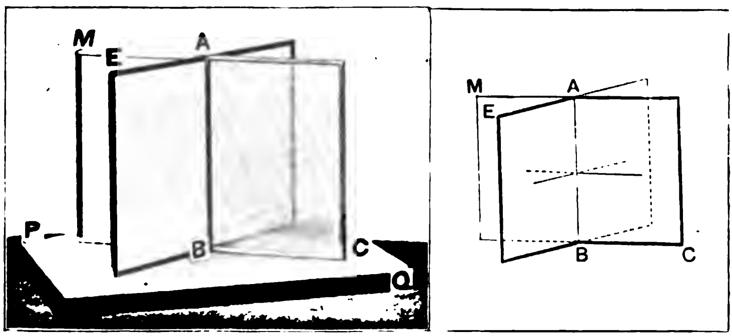
Thus, in the figure, the two planes BC and BD meeting in the line AB form the diedral angle $CABD$; BC and BD are the faces of the diedral angle, and AB is its edge.

If there is only one diedral angle at an edge, it may be designated by two points on its edge; thus the diedral angle $CABD$ may also be called the diedral angle AB.

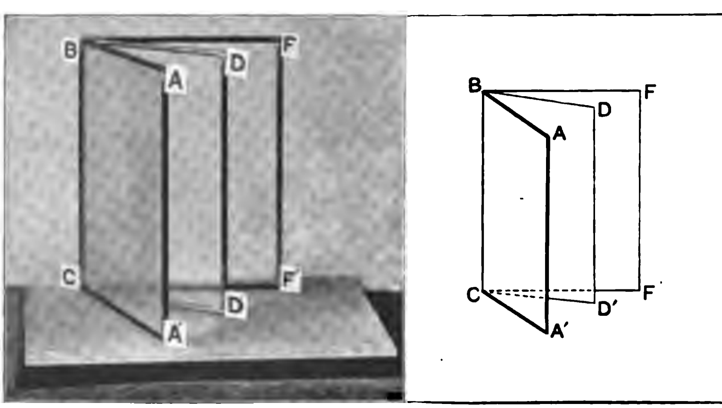
547. *Def.*—The **plane angle** of a diedral angle is the angle formed by two straight lines drawn one in each face of the diedral angle perpendicular to its edge at the same point.

Thus HKL is the plane angle of the diedral angle $CABD$.

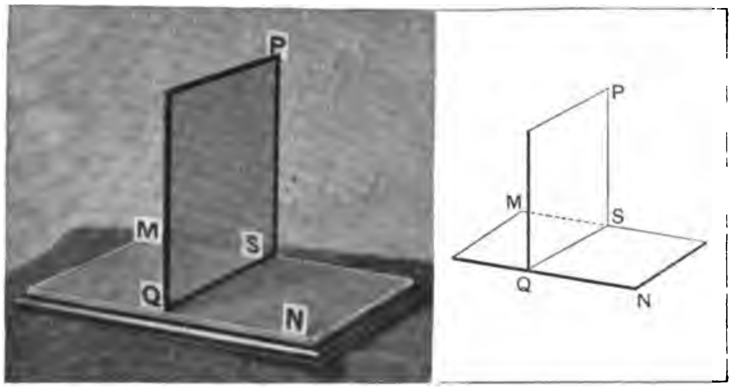
548. *Def.*—Two diedral angles are **vertical** if the faces of one are the prolongations of the faces of the other.



549. *Def.*—Two diedral angles are **adjacent** if they have a common edge and a common face lying between them; as $ABCD$ and $FBCD$.



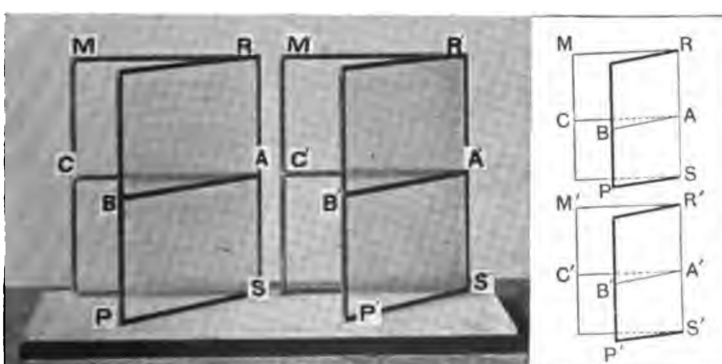
550. *Def.*—If a plane meets another plane so as to form with it two equal adjacent diedral angles, each of these diedral angles is a **right diedral angle**, and the first plane is **perpendicular** to the second.



Thus the plane PQ is perpendicular to the plane MN, if the diedral angles $PQSN$ and $PQSM$ are equal.

PROPOSITION XV. THEOREM

551. *If two diedral angles are equal, their plane angles are equal.*



GIVEN the equal diedral angles $MRSP$ and $M'R'S'P'$.

TO PROVE their plane angles CAB and $C'A'B'$ equal.

Place the diedral angle $M'R'S'P'$ in coincidence with its equal $MRSP$, making A' fall upon A.

Then, since $A'B'$ and AB are both in the plane RP and are perpendicular to the line RS at A, they coincide. § 18

Similarly $A'C'$ and AC coincide.

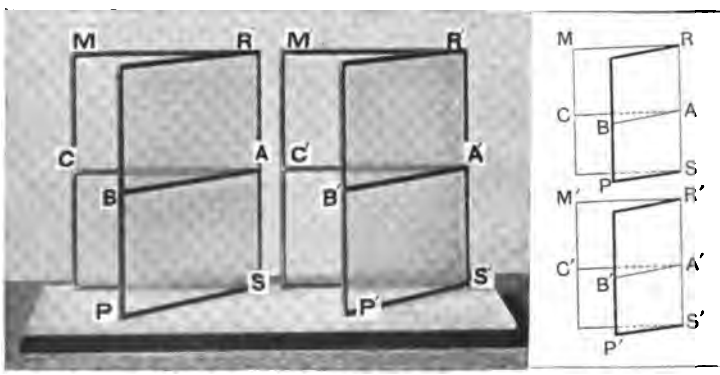
Therefore the angles CAB and $C'A'B'$ are equal. § 15

Q. E. D.

PROPOSITION XVI. THEOREM

552. *If the plane angles of two diedral angles are equal, the diedral angles are equal.*

[Converse of Proposition XV.]



GIVEN—two diedral angles, $MRSP$ and $M'R'S'P'$, whose plane angles, CAB and $C'A'B'$, are equal.

TO PROVE the diedral angles equal.

Since RS is perpendicular to the lines AB and AC, it is perpendicular to their plane. § 511

Similarly $R'S'$ is perpendicular to the plane of $A'B'$ and $A'C'$.

Place the angle $C'A'B'$ in coincidence with its equal CAB.

Then $R'S'$ will coincide with RS. § 515

Hence the planes $R'P'$ and RP will coincide. § 506

Similarly the planes $M'S'$ and MS will coincide.

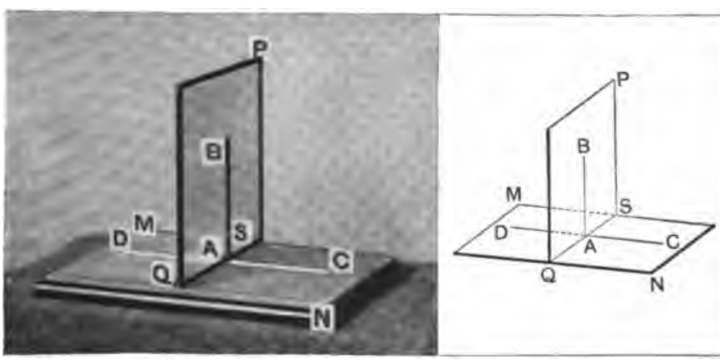
The diedral angles are therefore equal. § 15

Q. E. D.

553. COR. *Two vertical diedral angles are equal.*

PROPOSITION XVII. THEOREM

554. *If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the plane.*



GIVEN—the straight line AB perpendicular to the plane MN at A, and the plane PQ passed through AB intersecting MN in QS.

TO PROVE PQ perpendicular to MN.

Through A draw in MN the line CD perpendicular to QS.

Since AB is perpendicular to MN, it is perpendicular to QS and CD in MN. § 510

Hence BAC and BAD are right angles, and are the plane angles of the diedral angles $PQSN$ and $PQSM$. §§ 16, 547

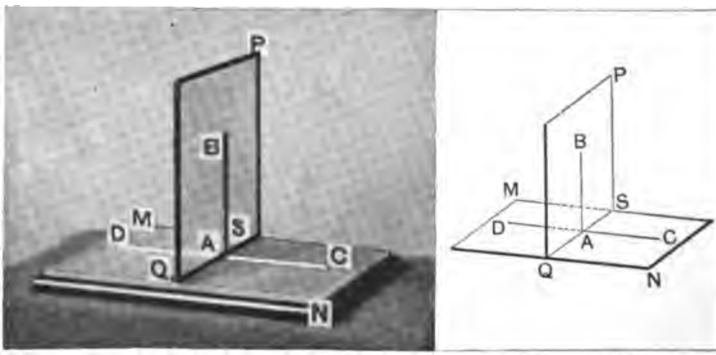
Therefore these diedral angles are equal. § 552

That is, PQ is perpendicular to MN. § 550

Q. E. D.

PROPOSITION XVIII. THEOREM

555. *If two planes are perpendicular to each other, a straight line drawn in one, perpendicular to their intersection, is perpendicular to the other.*



GIVEN—the plane PQ perpendicular to the plane MN and intersecting MN in QS. Draw AB in PQ perpendicular to QS at A.

TO PROVE AB perpendicular to MN.

Through A draw in MN the line CD perpendicular to QS.

Then BAC and BAD will be the plane angles of the equal diedral angles $PQSN$ and $PQSM$. § 547

Hence angle BAC = angle BAD. § 551

Therefore AB is perpendicular to CD. § 16

Since AB is perpendicular to CD and also to QS, it is perpendicular to MN. § 511

Q. E. D.

556. COR. I. *If two planes are perpendicular to each other, a straight line perpendicular to one at any point of their intersection lies in the other.*

Hint.—In the foregoing figure let AB be drawn perpendicular to MN at the point A of QS.

Then draw AB' in PQ perpendicular to QS.

Prove that AB and AB' coincide.

557. COR. II. *If two planes are perpendicular to each other, a straight line from any point of one perpendicular to the other lies in the first.*

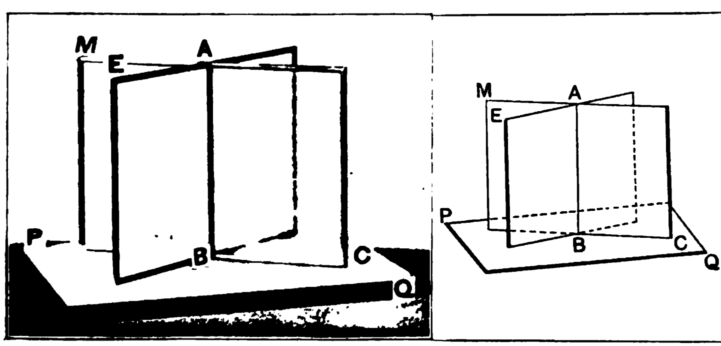
Hint.—In the foregoing figure let BA be drawn perpendicular to MN from the point B in PQ.

Then draw BA' in PQ perpendicular to QS.

Prove that BA and BA' coincide.

PROPOSITION XIX. THEOREM

558. *If two intersecting planes are perpendicular to a third plane, their intersection is perpendicular to that plane.*



GIVEN—the planes MC and EB perpendicular to the plane PQ and intersecting in AB.

TO PROVE AB perpendicular to PQ.

Through any point of AB draw a straight line perpendicular to PQ.

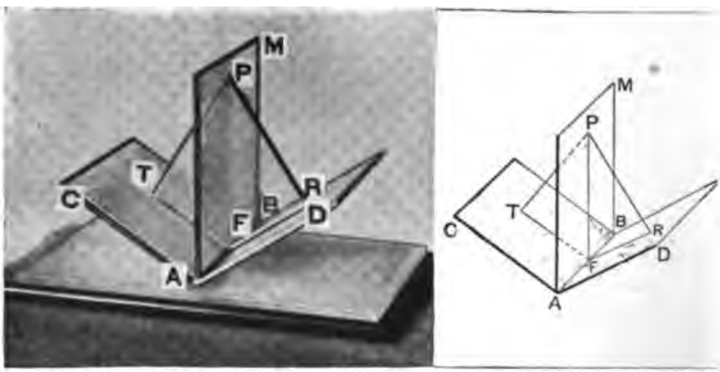
This line will lie in both MC and EB. §§ 556, 557

It must therefore coincide with their intersection AB.

Therefore AB is perpendicular to PQ. Q. E. D.

PROPOSITION XX. THEOREM

559. *Every point in the plane which bisects a diedral angle is equally distant from the faces of that angle.*



GIVEN—the plane MA bisecting the diedral angle $DABC$. Let P be any point in MA, and let PT and PR be the perpendiculars let fall from P to the faces BC and BD of the diedral angle.

TO PROVE $PT=PR$.

Through PT and PR pass a plane intersecting the planes BC, BD, and MA in FT, FR, and FP respectively.

Since the line PT is perpendicular to the plane BC, the plane PRT is perpendicular to the plane BC. § 554

Similarly the plane PRT is perpendicular to the plane BD.

Therefore PRT is perpendicular to their intersection AB. § 558

Hence AB is perpendicular to FT, FP, and FR. § 510

Hence PFT and PFR are the plane angles of the equal diedral angles $MABC$ and $MABD$. § 547

Therefore angle PFT = angle PFR. § 551

Consequently the right triangles PTF and PRF are equal. § 84

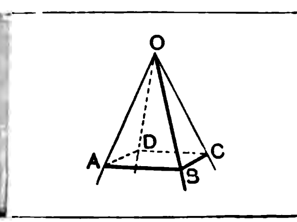
Therefore $PT = PR$. Q. E. D.

560. COR. *The locus of all points within a diedral angle equally distant from its faces is the plane which bisects the diedral angle.*

Hint.—It has been proved that all points in the bisecting plane possess the required property. It only remains to prove that any point outside does not, or that any point which possesses the required property must lie in AM. Let P' be such a point. Pass a plane through P' and the edge AB, and make constructions analogous to those in the preceding figure. Then prove that the plane $P'AB$ must bisect the diedral angle.

POLYEDRAL ANGLES

561. *Defs.*—When three or more planes meet in a point, they form a **polyedral angle.**



Thus the planes AOB, BOC, COD, DOA passing through the common point O form the polyedral angle $O-ABCD$.

The common point O is the **vertex** of the polyedral angle; the planes AOB, BOC, etc., are the **faces**; the intersections OA, OB, etc., of the faces are the **edges**; the angles AOB, BOC, etc., are the **face angles** of the polyedral angle.

The faces of a polyedral angle are supposed to be indefinite in extent. In order to show clearly in a figure the relative position of the edges, they are represented as being cut by a plane, as AC.

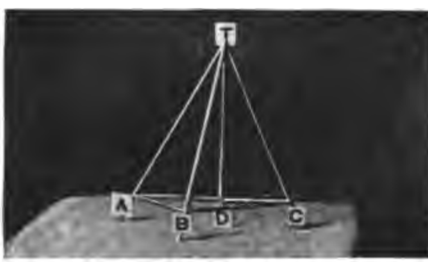
562. *Def.*—If a plane cuts all the edges of a polyedral angle the polygon formed by its intersection with the faces is a **section** of the polyedral angle.

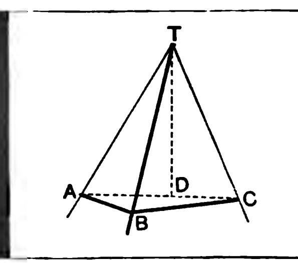
563. *Def.*—The face angles and the diedral angles formed by the faces are the **parts** of the polyedral angle.

564. *Def.*—A polyedral angle of three faces is a **triedral angle.**

PROPOSITION XXI. THEOREM

565. *The sum of any two face angles of a triedral angle is greater than the third.**





GIVEN—the triedral angle $T-ABC$ in which the face angle ATC is greater than either ATB or BTC.

TO PROVE $ATB+BTC>ATC$.

In the face ATC draw TD, making the angle $ATD=ATB$.

Take $TB=TD$, and through B and D pass a plane cutting the three faces in AB, BC, and AC.

The triangles ATB and ATD are equal. § 78

Hence $AB=AD$.

But $AB+BC>AC$.

By subtraction $BC>DC$.

* The theorem requires proof only when the third angle is greater than each of the others.

The triangles BTC and DTC have two sides of one equal to two sides of the other, and the third side BC of one greater than the third side DC of the other.

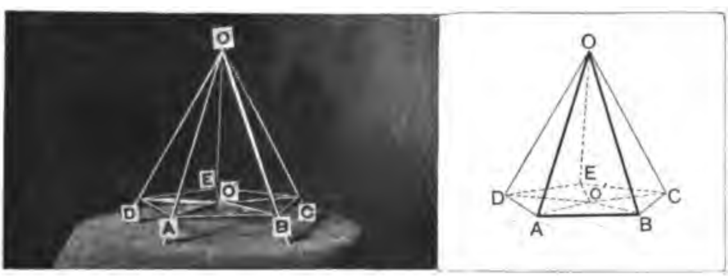
Therefore angle $BTC >$ angle DTC. § 92

By construction $ATB = ATD$.

Adding $ATB + BTC > ATC$. Q. E. D.

566. *Def.*—A polyedral angle is **convex** when any section by a plane forms a convex polygon.

567. *Exercise.*—The sum of the face angles of any convex polyedral angle is less than four right angles.



Hint.—Join any point O of the convex polygon $ABCED$ to its vertices.

In the triedral angles A, B, etc.,

$OAD \times OAB > DAB$; $OBA \times OBC > ABC$, etc. § 565

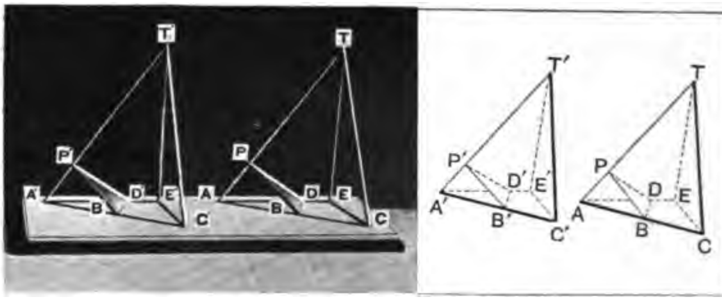
Adding these inequalities we get: the sum of the base angles of the triangles (AOB, etc.) about O is greater than the sum of the base angles of the triangles ($AO'B$, etc.) about O'.

But there are the same number of triangles about O as about O', so that the sum of all the angles of the triangles is the same in each case.

Hence the sum of the vertex angles at O is less than the sum of the vertex angles at O', i. e., less than four right angles. Ax. 6

PROPOSITION XXIV. THEOREM

568. *Two triedral angles are equal, if the three face angles of one are respectively equal to the three face angles of the other, provided the equal face angles are arranged in the same order.*



GIVEN—the triedral angles $T-ACE$ and $T'-A'C'E'$ having their face angles respectively equal and arranged in the same order.

TO PROVE— $T-ACE=T'-A'C'E'$.

On the six edges take equal lengths $TA=TC=TE=T'A'=T'C'=T'E'$ and join AC, CE, EA, $A'C'$, $C'E'$, $E'A'$.

Triangle ATC = triangle $A'T'C'$. § 78

Hence $AC=A'C'$, similarly $CE=C'E'$ and $EA=E'A'$.

Therefore triangle ACE = triangle $A'C'E'$. § 88

At any point P of TA draw PB, PD in the faces ATC, ATE respectively, perpendicular to TA.

PB must meet AC upon the side of A towards C, and PD must meet AD upon the side of A towards D. § 57

Join B and D, the points of meeting.

Repeat the construction in the triedral angle T', taking $A'P'=AP$.

Right triangle APB = right triangle $A'P'B'$. § 85

Therefore $AB=A'B'$ and $PB=P'B'$.

Similarly $AD=A'D'$ and $PD=P'D'$.

Hence triangle BAD = triangle $B'A'D'$. § 78

Hence $BD=B'D'$.

Therefore triangle BPD = triangle $B'P'D'$. § 88

Hence angle BPD = angle $B'P'D'$, that is, the plane angles of the diedral angles TA and $T'A'$ are equal. § 547

Therefore the diedral angles TA and $T'A'$ are equal. § 552

Place the diedral angle $T'A'$ in coincidence with diedral angle TA so that T' coincides with T.

Then $T'C'$ will coincide with TC and $T'E'$ with TE. Hyp.

The faces $E'T'C'$ and ETC will coincide. § 506, III

Therefore the triedral angles coincide and are equal.

Q. E. D.

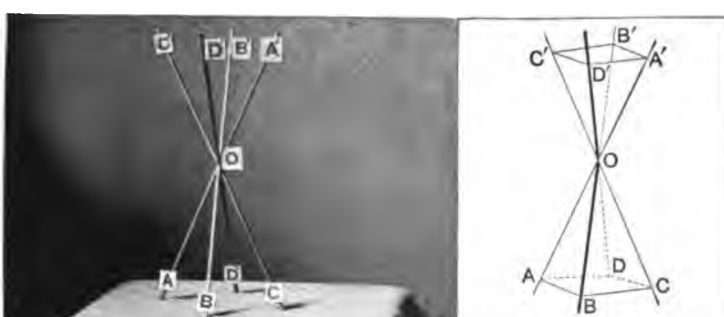
569. *Def.*—Two polyedral angles are **vertical,** if the edges of one are the prolongations, through the vertex, of the edges of the other.

570. *Def.*—Two polyedral angles are **symmetrical,** if all the parts of one are equal to those of the other, but arranged in opposite order.

Symmetrical polyedral angles are not in general equal, that is, cannot be made to coincide, just as we cannot put a right glove on the left hand.

PROPOSITION XXV. THEOREM

571. *Two vertical polyedral angles are symmetrical.*



GIVEN—the vertical polyedral angles $O—ABCD$ and $O—A'B'C'D'$.

TO PROVE them symmetrical.

The lines OA', OB', etc., are the prolongations of the lines OA, OB, etc., respectively.

Therefore the angles $A'OB'$, $B'OC'$, etc., are equal respectively to the angles AOB, BOC, etc. § 30

The planes $A'OB'$, $B'OC'$, etc., are the prolongations of the planes AOB, BOC, etc., respectively. § 506, III

Hence the diedral angles OA', OB', etc., are equal respectively to the diedral angles OA, OB, etc. § 553

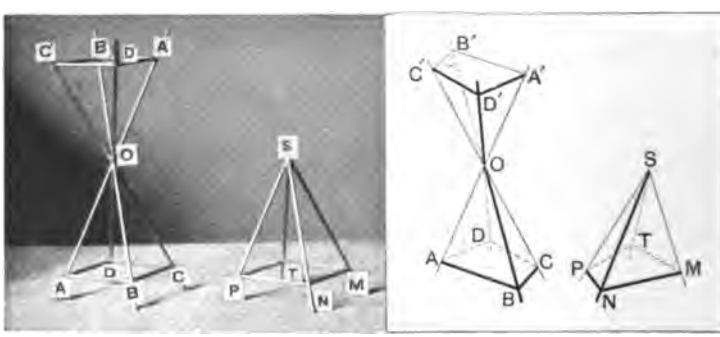
But the equal parts of the two polyedral angles are arranged in opposite order.*

Therefore they are symmetrical. § 570

[Having all the parts of one equal to those of the other, but arranged in opposite order.] Q. E. D.

PROPOSITION XXVI. THEOREM

572. *Either of two symmetrical polyedral angles is equal to the vertical of the other.*



GIVEN — two symmetrical polyedral angles, $O-ABCD$ and $S-MNPT$, the edges SM, SN, SP, ST corresponding to the edges OA, OB, OC, OD.

* A convenient way of seeing this is to conceive the eye placed at O. Then, if we look at the points $A'B'C'D'$, we find that they follow each other in an order of rotation in the same direction as the hand of a clock moves. This order is called "clockwise." But if we look at $ABCD$, still keeping the eye at O, the order $ABCD$ is "counter-clockwise."

TO PROVE—that $S-MNPT$ can be made to coincide with $O-A'B'C'D'$, the vertical of $O-ABCD$.

The parts of $S-MNPT$ and $O-ABCD$ are equal each to each and arranged in opposite order. § 570

Also the parts of $O-A'B'C'D'$ and $O-ABCD$ are equal each to each and arranged in opposite order. § 571

[Two vertical polyedral angles are symmetrical.]

Therefore the parts of $S-MNPT$ and $O-A'B'C'D'$ are equal each to each and arranged in the same order.

Place the polyedral angle $S-MNPT$ so that its diedral angle SM shall coincide with the equal diedral angle OA', the point S falling at O.

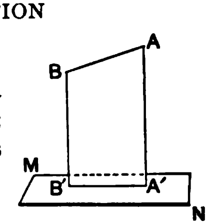
Since the parts of the two polyedral angles are arranged in the same order, the angles NSM and $B'OA'$ will then lie in the same plane.

Since they are equal, SN will coincide with OB'.

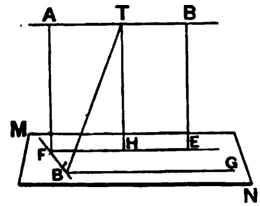
Similarly it can be proved that the next edge SP will coincide with OC' and so on until all the edges and therefore all the faces coincide.

Hence the polyedral angles $S-MNPT$ and $O-A'B'C'D'$ coincide and are equal. Q. E. D.

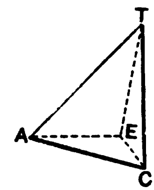
PROBLEMS OF DEMONSTRATION



573. *Exercise.*—Through any straight line a plane can be passed perpendicular to any plane; and only one such plane can be passed unless the given line is itself perpendicular to the given plane.

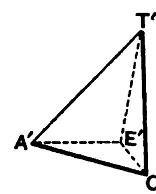


574. *Exercise.* — Between two straight lines not in the same plane a common perpendicular can be drawn, and only one.



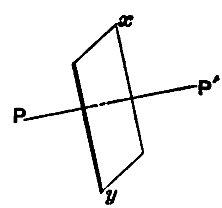
575. *Exercise.*—Two triedral angles are equal, if two face angles and the included diedral angle of one are respectively equal to two face angles and the included diedral angle of the other, the parts given equal being arranged in the same order.

[Where is this proved in Prop. XXIV?]

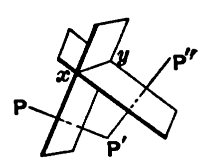


576. *Exercise.*—Two triedral angles are equal, if two diedral angles and the included face angle of one are respectively equal to two diedral angles and the included face angle of the other, the parts given equal being arranged in the same order.

LOCI



577. *Exercise.* — Find, and prove correct, the locus of all points in space equidistant from two given points.

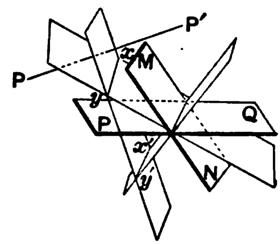


578. *Exercise.*—Find, and prove correct, the locus of all points equidistant from three given points.

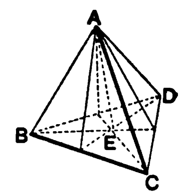
579. *Exercise.* — Find the locus of points at a given distance from a given plane.

580. *Exercise.*—Find the locus of points equidistant from two parallel planes.

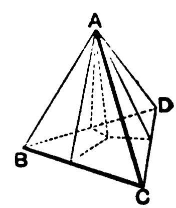
581. *Exercise.*—Find the locus of points equidistant from two intersecting planes.



582. *Exercise.*—Find, and prove correct, the locus of points which are equidistant from two given planes, and at the same time equidistant from two given points.



583. *Exercise.* — In any triedral angle the three planes bisecting the three diedral angles intersect in a common straight line, which is the locus of points within the triedral angle equidistant from its faces.



584. *Exercise.* — In any triedral angle the three planes passed through the bisectors of the three face angles, and perpendicular to these faces respectively, intersect in a common straight line, every point of which is equidistant from the edges of the triedral angle.

PROBLEMS OF CONSTRUCTION

The constructions of solid geometry differ from those of plane geometry in that we cannot perform them with ruler and compasses, or with any instruments of drawing.

We shall therefore consider a problem of construction in solid geometry solved when it is reduced to one or more of the following elementary constructions which we assume can be performed, viz.:

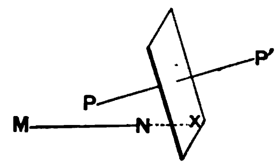
(1.) A plane can be passed through any three given points.

(2.) The intersection of a plane with any given straight line or with any given plane can be determined.

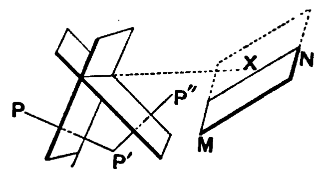
(3.) A straight line can be drawn through any given point perpendicular to any given plane.

585. *Exercise.*—Determine a point in a given straight line which shall be equidistant from two given points in space.

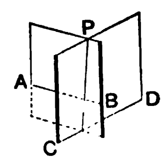
Do not assume that the given line and the given points are in the same plane, and avoid similar assumptions in the following exercises.



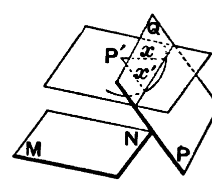
586. *Exercise.* — Determine a point in a plane MN which shall be equidistant from three given points in space, P, P', and P''.



587. *Exercise.*—Through a given point P in space determine a straight line which shall cut two given straight lines AB and CD.

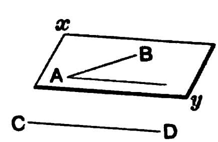


588. *Exercise.*—Given a point P' and any two non-parallel planes MN and PQ. From the point draw a straight line of given length terminating in one of the planes and parallel to the other.

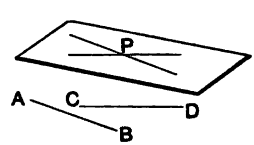


589. *Exercise.*—Show how to pass a plane through a straight line AB parallel to another straight line CD.

Hint.—Apply § 530.



590. *Exercise.*—Show how to pass a plane through a point P parallel to two given straight lines AB and CD.



GEOMETRY OF SPACE

BOOK VII

POLYEDRONS

591. *Defs.*—A **polyedron** is a geometrical solid bounded by planes.

The lines of intersection of the bounding planes are the **edges** of the polyedron; the points of intersection of the edges are the **vertices**; the portions of the bounding planes bounded by the edges are the **faces.**

The least number of faces that a polyedron can have is four; for three planes by intersecting form a triedral angle, and one more plane is necessary to enclose with these a definite portion of space.

592. *Defs.*—A polyedron of four faces is a **tetraedron**; one of six faces, a **hexaedron**; one of eight faces, an **octaedron**; one of twelve faces, a **dodecaedron**; one of twenty faces, an **icosaedron.**

ICOSAEDRON DODECAEDRON OCTAEDRON HEXAEDRON TETRAEDRON

593. *Def.*—A polyedron is **convex** when no face, if produced, will enter the polyedron.

All the polyedrons treated of in this book will be understood to be convex.

PRISMS. PARALLELOPIPEDS

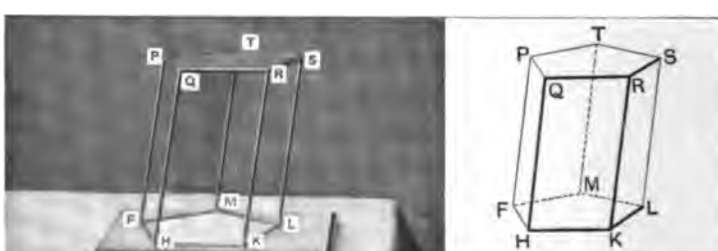
594. *Defs.*—A prism is a polyedron, two of whose faces (the **bases**) are equal polygons having their corresponding sides parallel; each of the remaining faces (the **lateral faces**) is formed by a plane passed through two corresponding sides of the bases.

An edge which is the intersection of two lateral faces is a **lateral edge.**

PRISMS

PROPOSITION I. THEOREM

595. *The lateral faces of a prism are parallelograms.*



GIVEN the prism FS.

TO PROVE its lateral faces are parallelograms.

QR is equal and parallel to HK. § 594

Therefore $HQRK$ is a parallelogram. § 123

Similarly the other lateral faces are parallelograms. Q. E. D.

596. COR. I. *The lateral edges of a prism are parallel.*

597. COR. II. *The lateral edges of a prism are equal.*

598. *Defs.*—A **right prism** is a prism whose lateral edges are perpendicular to its bases.

An **oblique prism** is a prism whose lateral edges are not perpendicular to its bases.

599. *Def.*—A **regular prism** is a right prism whose bases are regular polygons.

600. COR. III. *The lateral faces of a right prism are rectangles.*

601. *Def.*—A **parallelopiped** is a prism whose bases are parallelograms.

602. *Def.*—A **right parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases.

OBLIQUE PARALLELOPIPED — RIGHT PARALLELOPIPED — RECTANGULAR PARALLELOPIPED — CUBE

603. *Def.*—A **rectangular parallelopiped** is a right parallelopiped whose bases are rectangles.

604. *Def.*—A **cube** is a right parallelopiped whose bases are squares and whose lateral edges are equal to the sides of its base.

605. COR. IV. *All the faces of a parallelopiped are parallelograms.*

606. COR. V. *All the faces of a rectangular parallelopiped are rectangles.*

607. COR. VI. *All the faces of a cube are equal squares.*

608. *Def.*—The lateral faces of a prism form a **prismatic surface.**

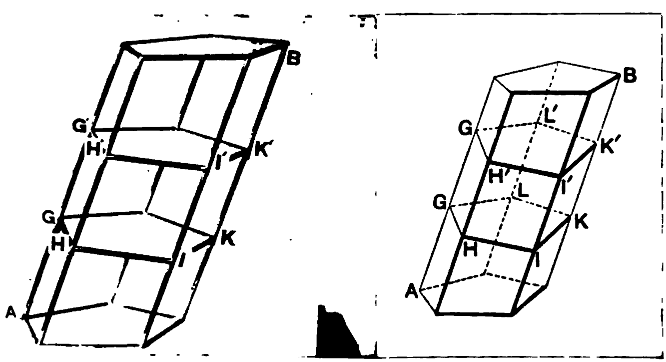
This prismatic surface may extend beyond the bases of the prism.

Discussion.—A prismatic surface can be generated by a moving straight line which continually intersects the perimeter of a given fixed polygon and is constantly parallel to a given fixed straight line.

For a straight line intersecting one side of the base of a prism and moving so as to remain parallel to the lateral edges will generate a lateral face.

PROPOSITION II. THEOREM

609. *The sections of a prismatic surface made by two parallel planes cutting its edges are equal polygons.*



GIVEN—the prismatic surface AB cut by two parallel planes in the sections $GHIKL$ and $G'H'I'K'L'$.

TO PROVE these polygons are equal.

The sides GH, HI, etc., are parallel respectively to $G'H'$, $H'I'$, etc. § 524

Hence $GH=G'H'$, $HI=H'I'$, etc. § 116

Also angle $GHI=$angle $G'H'I'$,

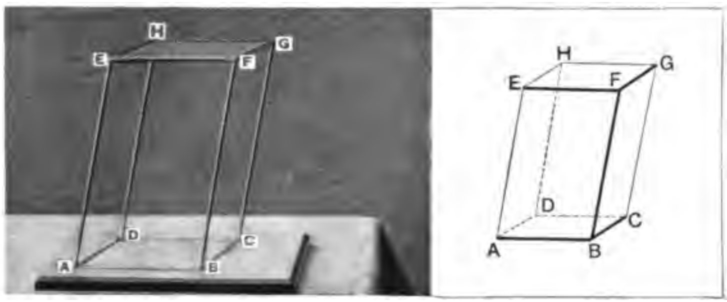
angle $HIK=$angle $H'I'K'$, etc. § 537

Therefore the polygons $GHIKL$ and $G'H'I'K'L'$ are mutually equilateral and equiangular.

Hence they can be made to coincide and are equal. Q. E. D.

PROPOSITION III. THEOREM

610. *Any two opposite faces of a parallelopiped may be taken as its bases.*



GIVEN—the parallelopiped AG, the bases being first taken as AC and EG.

TO PROVE—that any other two opposite faces, as AF and DG, may be taken as bases.

EF is parallel and equal to HG, FB to GC, etc. §§ 601, 596

Also angle EFB = angle HGC, etc. § 537

Hence parallelograms $AEFB$ and $DHGC$ can be made to coincide and are equal.

These parallelograms also have their corresponding sides parallel.

Therefore they may be taken as bases. §§ 595, 601

Q. E. D.

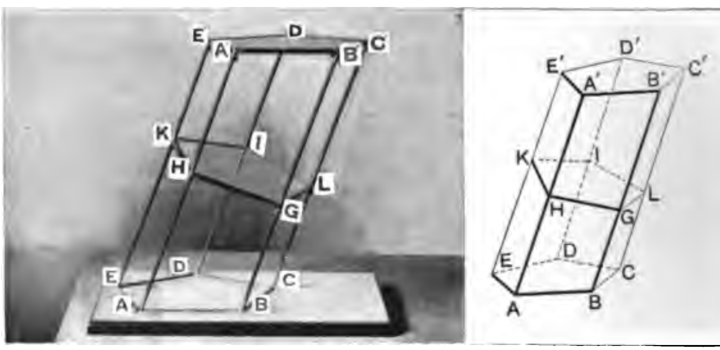
611. *Def.*—A **right section** of a prism is the section formed by a plane perpendicular to the lateral edges.

612. *Def.*—The **lateral area** of a prism is the sum of the areas of its lateral faces.

613. *Def.*—The **altitude** of a prism is the perpendicular distance between the planes of its bases.

PROPOSITION IV. THEOREM

614. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*



GIVEN—the prism AC', of which $HGLIK$ is a right section.
TO PROVE—its lateral area $=(HG+GL+\text{etc.})\times AA'$.

The lateral area is the sum of the areas of the lateral faces, which are parallelograms. § 595

The area of each parallelogram is equal to the product of its base and altitude. § 366

Their bases AA', BB', etc., are all equal. § 597

Their altitudes are the lines HG, GL, etc. § 510

Hence by addition

$$\text{lateral area} = (HG+GL+\text{etc.})\times AA'. \qquad \text{Q. E. D.}$$

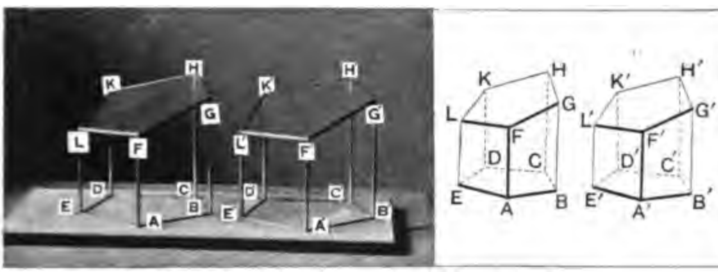
615. COR. *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

616. *Defs.*—A **truncated prism** is a polyedron bounded by a prismatic surface and two faces (the **bases**), whose planes are not parallel.

617. *Def.*—A truncated prism is **right** when one of its bases is perpendicular to the lateral edges.

PROPOSITION V. THEOREM

618. *Two right truncated prisms are equal, if three lateral edges of one are equal to the three corresponding edges of the other and the bases to which they are respectively perpendicular are equal.*



GIVEN—the right truncated prisms AK and $A'K'$, having the lateral edges AF and $A'F'$, BG and $B'G'$, CH and $C'H'$ respectively equal, and perpendicular to the equal bases $ABCDE$, $A'B'C'D'E'$.

TO PROVE that AK and $A'K'$ are equal.

Superpose the truncated prisms so that the bases $ABCDE$ and $A'B'C'D'E'$ shall coincide.

Then the indefinite lines AF, BG, etc., coincide respectively with $A'F'$, $B'G'$, etc. § 515

Hence the indefinite prismatic surfaces coincide. § 506 IV

Since $AF = A'F'$, F falls on F'. Similarly G falls on G' and H upon H'.

Hence the planes of the upper bases coincide. § 506 I

Therefore the truncated prisms coincide and are equal.

Q. E. D.

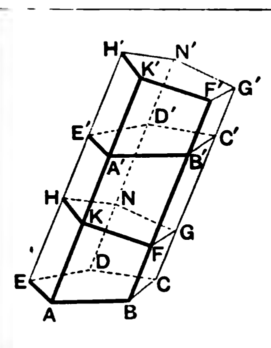
619. COR. *Two right prisms are equal, if they have equal bases and equal altitudes.*

620. *Defs.*—The **volume** of any solid is its ratio to another solid taken arbitrarily as the **unit of volume.**

621. *Def.*—Two solids are **equivalent** when their volumes are equal.

PROPOSITION VI. THEOREM

622. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.*



GIVEN—the oblique prism $ABCDE\text{-}A'$ of which $KFGNH$ is a right section.

Produce AA' to K', making $KK'=AA'$, and through K' pass a plane parallel to $KFGNH$, cutting the other edges produced in F', G', N', H'.

TO PROVE—the oblique prism $ABCDE\text{-}A'$ is equivalent to the right prism $KFGNH\text{-}K'$.

The truncated right prisms AG and $A'G'$ have the bases $KFGNH$ and $K'F'G'N'H'$ equal. § 609

Also the lateral edges AK, BF, and CG are respectively equal to $A'K'$, $B'F'$, and $C'G'$. Ax. 3

Therefore these truncated prisms are equal. § 618

If the upper truncated prism $A'G'$ be taken from the whole figure, the oblique prism will be left.

If the lower truncated prism AG be taken from the whole figure, the right prism will be left.

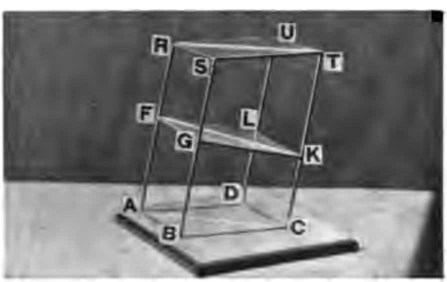
Therefore the oblique prism is equivalent to the right prism. Ax. 3

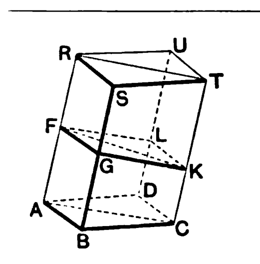
Q. E. D.

623. *Defs.*—A **triangular prism** is one whose base is a triangle; a **quadrangular,** one whose base is a quadrilateral.

PROPOSITION VII. THEOREM

624. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*





GIVEN—the parallelopiped $ABCD\text{-}R$ divided by the plane $ARTC$ into two triangular prisms $ABC\text{-}S$ and $ACD\text{-}U$.

TO PROVE these triangular prisms are equivalent.

Let $FGKL$ be a right section of the parallelopiped, cutting the plane $ARTC$ in FK.

The planes AU and BT are parallel. § 531

Therefore FL and GK are parallel. § 524

Similarly FG and LK are parallel.

Therefore $FGKL$ is a parallelogram. § 112

Hence the triangles FGK and FKL are equal. § 114

The triangular prism $ABC\text{-}S$ is equivalent to a right prism whose base is the right section FGK and whose altitude is the lateral edge AR, and $ACD\text{-}U$ is equivalent to a right prism whose base is FKL and whose altitude is AR. § 622

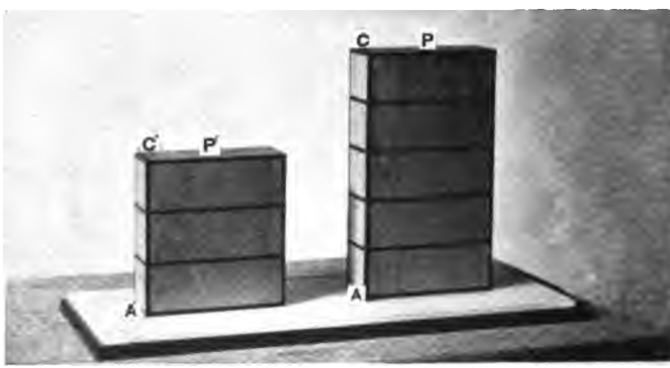
These two right prisms are equal. § 619

Therefore $ABC\text{-}S$ and $ACD\text{-}U$ are equivalent. Ax. 1

Q. E. D.

PROPOSITION VIII. THEOREM

625. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.**



GIVEN—the rectangular parallelopipeds P and P' having equal bases, their altitudes being AC and $A'C'$.

TO PROVE $$\frac{P'}{P}=\frac{A'C'}{AC}.$$

CASE I. *When the altitudes are commensurable.*

Suppose the common measure of AC and $A'C'$ to be contained in AC 5 times, and in $A'C'$ 3 times.

Then $$\frac{A'C'}{AC}=\frac{3}{5}.$$

* The ratio of two polyedrons means the ratio of their volumes.

Through the points of division of AC and $A'C'$ pass planes parallel to the bases.

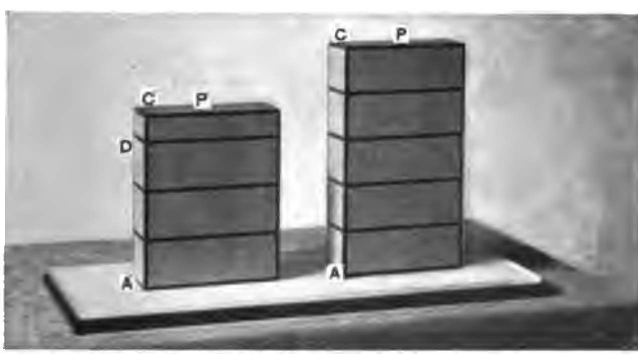
These planes divide the parallelopipeds into smaller parallelopipeds, all of which are equal. §§ 609, 619

P contains 5 and P' contains 3 of these small parallelopipeds.

Hence $$\frac{P'}{P} = \frac{3}{5}.$$

Therefore $$\frac{P'}{P} = \frac{A'C'}{AC}.$$ Ax 1.

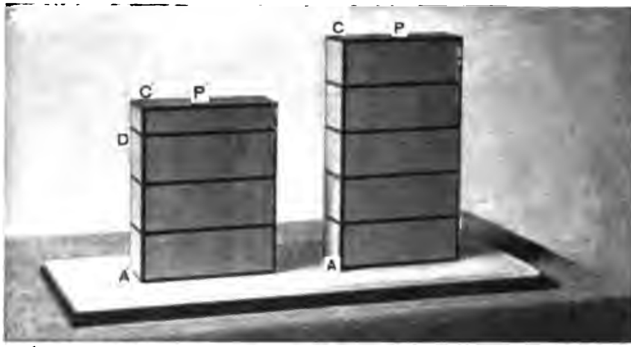
CASE II. *When the altitudes are incommensurable.*



Divide AC into any number of equal parts and apply one of these parts to $A'C'$ as often as $A'C'$ will contain it.

Since AC and $A'C'$ are incommensurable, there will be a remainder DC' less than one of these parts.

Pass a plane through D parallel to the bases of P' and let X be the rectangular parallelopiped between this plane and the *lower* base of P'.



Then, since $A'D$ and AC are *commensurable*,

$$\frac{X}{P}=\frac{A'D}{AC}. \qquad \text{Case I}$$

If each of the parts of AC be continually bisected, each part can be made as small as we please.

Therefore DC', which is always less than one of these parts, can be made as small as we please.

But it can never be reduced to zero, since AC and $A'C'$ are given incommensurable.

Therefore $A'D$ will approach $A'C'$ as a limit. § 181

Hence $\frac{A'D}{AC}$ will approach $\frac{A'C'}{AC}$ as a limit.

Likewise $\frac{X}{P}$ will approach $\frac{P'}{P}$ as a limit.

Therefore $$\frac{P'}{P}=\frac{A'C'}{AC}. \qquad \S\ 182$$

Q. E. D.

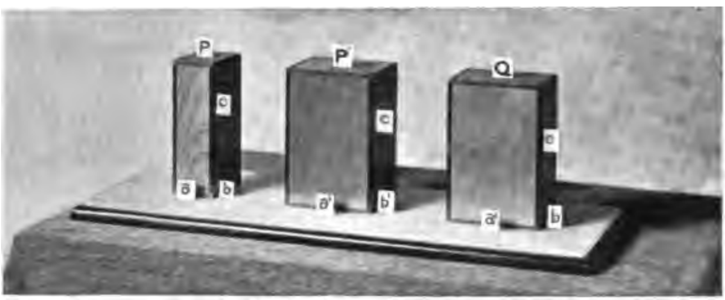
626. *Def.*—The three edges of a rectangular parallelopiped meeting at a common vertex are its **dimensions.**

627. *Remark.*—The preceding theorem may be stated thus:

Two rectangular parallelopipeds having two ***dimensions in common*** *are to each other as their third dimensions.*

PROPOSITION IX. THEOREM

628. *Two rectangular parallelopipeds which have one dimension in common are to each other as the products of the two other dimensions.*



GIVEN—the rectangular parallelopipeds P and P', having the dimension c common, the other dimensions being a, b and a', b' respectively.

TO PROVE $$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$

Let Q be a third rectangular parallelopiped having the dimensions a', b, c.

Then P and Q have two dimensions b and c in common.

Hence $$\frac{P}{Q} = \frac{a}{a'}.$$ § 627

Also Q and P' have two dimensions a' and c in common.

Hence $$\frac{Q}{P'} = \frac{b}{b'}.$$

Multiplying these equations together,

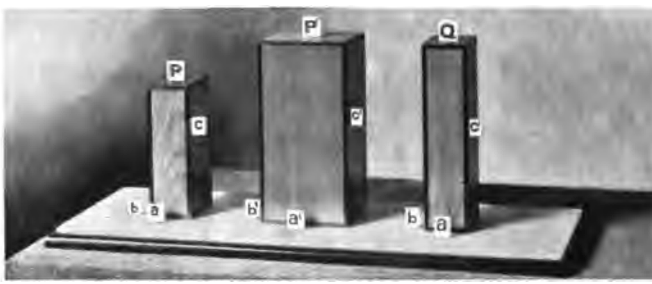
$$\frac{P}{P'} = \frac{a \times b}{a' \times b'}.$$ Q. E. D.

629. *Remark.*—This theorem may be stated thus:

Two rectangular parallelopipeds having equal altitudes are to each other as their bases.

PROPOSITION X. THEOREM

630. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



GIVEN—the rectangular parallelopipeds P and P', whose dimensions are a, b, c and a', b', c' respectively.

TO PROVE $$\frac{P}{P'}=\frac{a \times b \times c}{a' \times b' \times c'}.$$

Let Q be a third rectangular parallelopiped having the dimensions a, b, c'.

Then $$\frac{P}{Q}=\frac{c}{c'}. \qquad \S\ 627$$

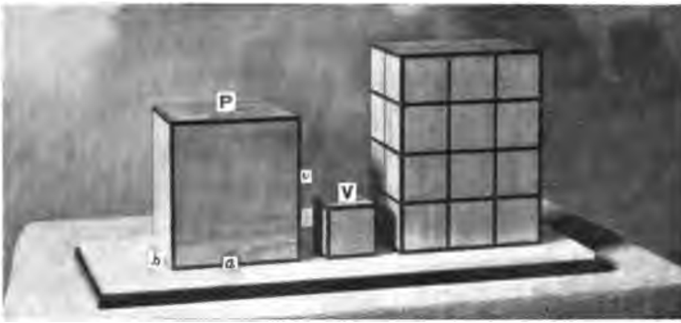
And $$\frac{Q}{P'}=\frac{a \times b}{a' \times b'}. \qquad \S\ 628$$

Multiplying these equations together,

$$\frac{P}{P'}=\frac{a \times b \times c}{a' \times b' \times c'}.$$ Q. E. D.

PROPOSITION XI. THEOREM

631. *The volume of a rectangular parallelopiped is equal to the product of its three dimensions, provided the unit of volume is a cube whose edge is the linear unit.*



Proof.—Let P be any rectangular parallelopiped whose dimensions are a, b, and c, and let the cube V, whose edge is the linear unit, be the unit of volume.

Then $\frac{P}{V}$ is the volume of P. § 620

But $\frac{P}{V}=\frac{a\times b\times c}{1\times 1\times 1}=a\times b\times c.$ § 630

Therefore vol. $P=a\times b\times c.$ Q. E. D.

632. *Remark.*—Hereafter the unit of volume is to be understood to be a cube whose edge is the linear unit.

633. *Remark.*—This theorem may also be stated:

The volume of a rectangular parallelopiped is equal to the product of its base and altitude.

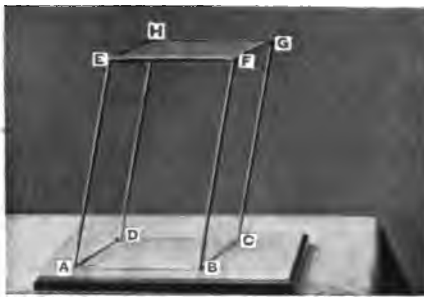
634. COR. *The volume of a cube is equal to the third power of its edge.*

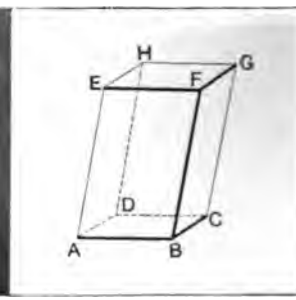
Hence it is that the *third power* of a number is called the *cube* of that number.

635. *Remark.*—When the three dimensions of a rectangular parallelopiped are exactly divisible by the linear unit, the truth of the proposition may be rendered evident by dividing the parallelopiped into cubes, whose edges are equal to the linear unit.

Thus, if three edges which meet at a common vertex are respectively 2 units, 3 units, and 4 units in length, the parallelopiped may be divided into 24 cubes, each equal to the unit of volume, by passing planes perpendicular to the edges through their points of division.

636. CONSTRUCTION. *To construct a parallelopiped having as edges three given straight lines drawn from the same point.*





GIVEN the straight lines AB, AD, and AE.

TO CONSTRUCT—a parallelopiped having them as edges.

Pass a plane through each pair of the given straight lines.

Then pass a plane through the extremity of each line parallel to the plane of the other two.

The solid thus formed may be shown to be a parallelopiped by applying successively §§ 524, 116, 537, 594, 601.

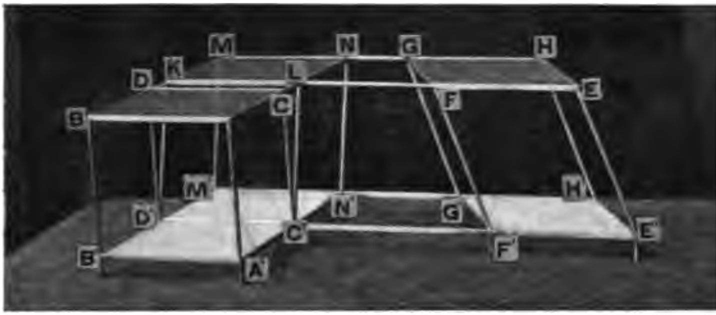
637. *Exercise.*—Show that if the three given lines in the preceding construction are perpendicular to each other, the parallelopiped formed will be rectangular.

PROPOSITION XII. THEOREM

638. *The volume of any parallelopiped is equal to the product of its base and altitude.*

GIVEN—the *oblique* parallelopiped FH', whose base is $F'E'H'G'$ and altitude h.

TO PROVE vol. $FH' = F'E'H'G' \times h$.



On the edge $E'F'$ produced take $C'D'=E'F'$.

Through C' and D' pass planes perpendicular to $E'D'$, forming the parallelopiped KN', which is a right parallelopiped if LN' and KM' are considered the bases.

On the edge $N'C'$ produced of the right parallelopiped KN' take $C'A'=N'C'$.

In the plane $A'N'$ draw $C'C$ perpendicular to $A'C'$.

The three lines $C'D'$, $C'A'$, $C'C$ are perpendicular to each other. § 510

Therefore the parallelopiped BC' constructed upon them as edges will be rectangular. § 637

The rectangular and oblique parallelopipeds are each equivalent to the right parallelopiped KN', and therefore to each other. § 622

Their bases $F'E'H'G'$ and $B'A'C'D'$ are each equivalent to $D'C'N'M'$, and therefore to each other. § 367

And the altitude of each is h. § 545

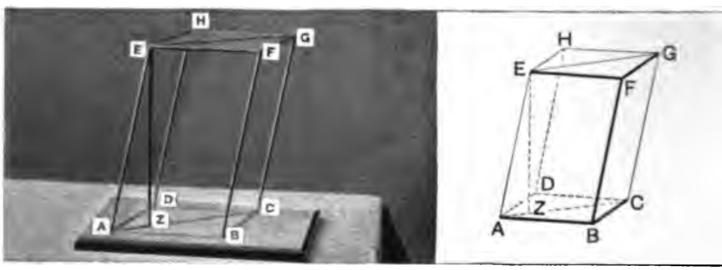
But vol. $BC'=B'A'C'D'\times h$. § 631

Therefore vol. $FH'=F'E'H'G'\times h$. Q. E. D.

639. *Exercise.*—The base of a parallelopiped is a rectangle 12 in. by 34 in.; the altitude of the parallelopiped is 17 in. Find its volume.

PROPOSITION XIII. THEOREM

640. *The volume of a triangular prism is equal to the product of its base and altitude.*



GIVEN—the triangular prism $ABC\text{-}F$ having the base ABC and altitude EZ.

TO PROVE vol. $ABC\text{-}F = ABC \times EZ$.

Construct the parallelopiped $ABCD\text{-}F$ having BA, BC, and BF as edges. § 636

Then the volume of the parallelopiped is equal to the product of its base $ABCD$ and its altitude EZ. § 638

But the volume of the triangular prism is half the volume of the parallelopiped; its base is half the base of the parallelopiped; and its altitude is the same. §§ 624, 114

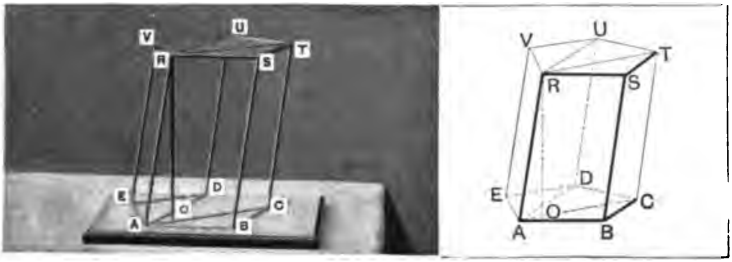
Therefore the volume of the triangular prism is equal to the product of its base ABC and its altitude EZ. Q. E. D.

PROPOSITION XIV. THEOREM

641. *The volume of any prism is equal to the product of its base and altitude.*

GIVEN—the prism $ABCDE\text{-}R$ with base $ABCDE$ and altitude RO.

TO PROVE vol. $ABCDE\text{-}R = ABCDE \times RO$.



The prism may be divided into triangular prisms by planes passed through AR and the diagonally opposite edges.

The volume of each triangular prism is equal to the product of its base and altitude. § 640

They have the common altitude RO. § 545

Therefore the volume of the whole prism is equal to the product of the sum of the bases of the triangular prisms, i. e., the base of the whole prism, and the common altitude.

Q. E. D.

642. COR. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

643. COR. II. *Any two prisms are to each other as the products of their bases and altitudes.*

Hint.—Prove as in § 368.

644. COR. III. *Two prisms having equivalent bases are to each other as their altitudes.*

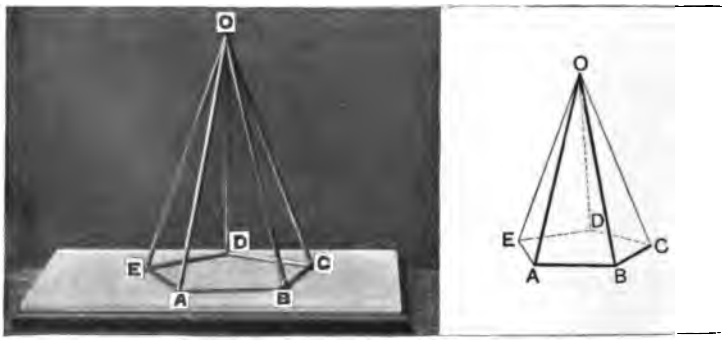
645. COR. IV. *Two prisms having equal altitudes are to each other as their bases.*

PYRAMIDS

646. *Defs.*—A **pyramid** is a polyedron one of whose faces is a polygon and whose other faces are formed by planes passed through the sides of the polygon and a common point without the plane of the polygon.

Hint.—The faces formed by these planes are triangles having the sides of the polygon as bases and a common vertex.

The polygon is the **base**; the triangles are the **lateral faces**; the common vertex of the triangles is the **vertex** of the pyramid; and the edges passing through the vertex are the **lateral edges.**

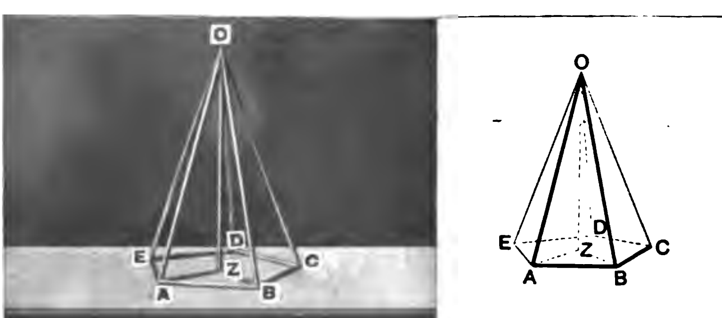


Thus *ABCDE* is the base; *O* is the vertex; *OA*, *OB*, etc., are the lateral edges; and *OAB*, *OBC*, etc., are the lateral faces of the pyramid *O-ABCDE*.

647. *Defs.*—A **regular pyramid** is a pyramid whose base is a regular polygon and whose vertex lies in the perpendicular to the base erected at its centre. This perpendicular is the **axis** of the regular pyramid.

PROPOSITION XV. THEOREM

648. *The lateral edges of a regular pyramid are equal.*



GIVEN the regular pyramid $O\text{-}ABCDE$.

TO PROVE $OA = OB = OC =$ etc.

Let OZ be the axis of the regular pyramid.

Then $ZA = ZB = ZC =$ etc. § 438 III

Therefore $OA = OB = OC =$ etc. § 519 I

Q. E. D.

649. COR. I. *The lateral faces of a regular pyramid are equal isosceles triangles.*

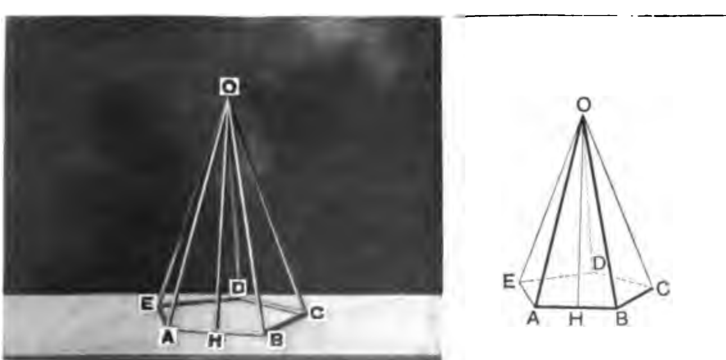
650. COR. II. *The altitudes of the lateral faces drawn from the common vertex O are equal.*

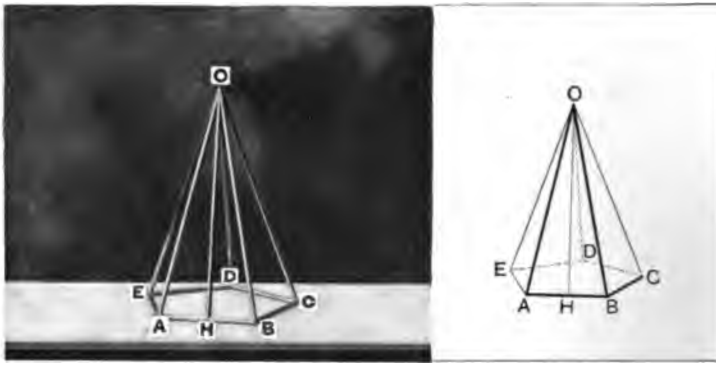
651. *Def.*—The **slant height** of a regular pyramid is the altitude of any one of its lateral faces drawn from the vertex of the pyramid.

652. *Def.*—The **lateral area** of a pyramid is the sum of the areas of its lateral faces.

PROPOSITION XVI. THEOREM

653. *The lateral area of a regular pyramid is equal to one-half the product of the perimeter of its base and its slant height.*





GIVEN—the regular pyramid $OABCDE$, of which OH is the slant height.

TO PROVE—lat. area $O\text{-}ABCDE=\frac{1}{2}(AB+BC+\text{etc.})\times OH$.

The lateral area of the pyramid is the sum of the areas of the triangles OAB, OBC, etc. § 652

The area of each triangle is equal to half the product of its base and altitude.

Hence area $OAB=\frac{1}{2}AB\times OH$,

area $OBC=\frac{1}{2}BC\times OH$, etc. § 650

Therefore the lateral area of the pyramid is

$\frac{1}{2}(AB+BC+\text{etc.})\times OH$. Q. E. D.

654. *Defs.*—A **truncated pyramid** is the portion of a pyramid contained between its base and a plane cutting all its lateral edges.

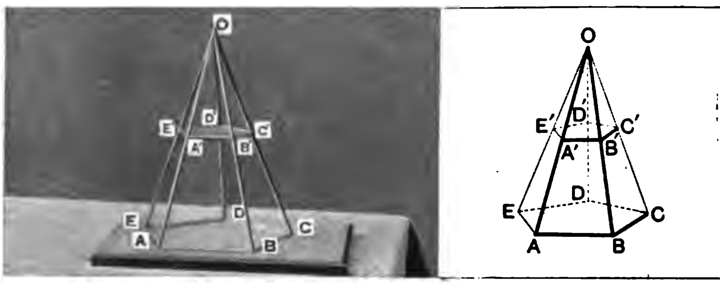
The section thus made and the base of the pyramid are the **bases** of the truncated pyramid.

The other faces are the **lateral faces** of the truncated pyramid.

655. *Def.*—A **frustum of a pyramid** is a truncated pyramid, the planes of whose bases are parallel.

PROPOSITION XVII. THEOREM

656. *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*



GIVEN—the frustum EC' of the regular pyramid $O\text{-}ABCDE$.
TO PROVE—its faces are equal trapezoids, viz.: $ABB'A' = BCC'B'$, etc.

The faces are trapezoids, since $A'B'$, $B'C'$, etc., are parallel to AB, BC, etc., respectively. § 524

Place the equal isosceles triangles OAB, OBC in coincidence by revolving the former about OB.

Then also must $A'B'$ coincide with $B'C'$, both passing through B' and being parallel to BC. Ax. *b*

Thus the two trapezoids coincide and are equal. Likewise all the trapezoids are equal. Q. E. D.

657. *Def.*—The **slant height** of a frustum of a regular pyramid is the altitude of any lateral face.

658. COR. *The lateral area of a frustum of a regular pyramid is equal to one-half the product of the sum of the perimeters of its bases and its slant height.*

Hint.—Apply § 375.

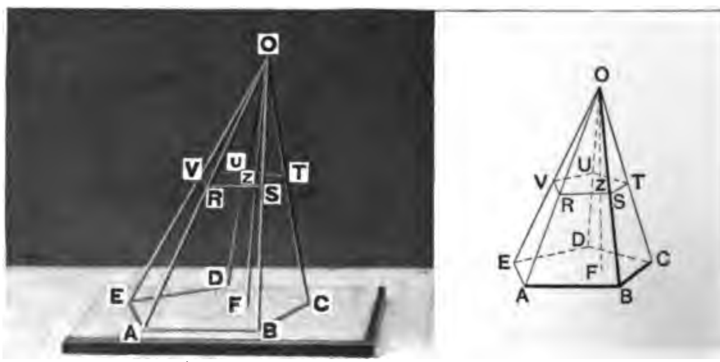
659. *Def.*—The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

PROPOSITION XVIII. THEOREM

660. *If a pyramid is cut by a plane parallel to its base:*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is a polygon similar to the base.*



GIVEN—the pyramid $O—ABCDE$ cut by a plane parallel to its base $ABCDE$ in the section $RSTUV$, and the altitude OF cutting the plane of the section in Z.

I. TO PROVE $\frac{OR}{OA}=\frac{OS}{OB}=\frac{OT}{OC}=\text{etc.}=\frac{OZ}{OF}\cdot$

This follows immediately from § 536.

II. TO PROVE $RSTUV$ is similar to $ABCDE$.

The corresponding sides of the two polygons are parallel. § 524

Hence their angles are equal. § 537

Also the triangles ORS, OST, etc., are similar to OAB, OBC, etc. § 262

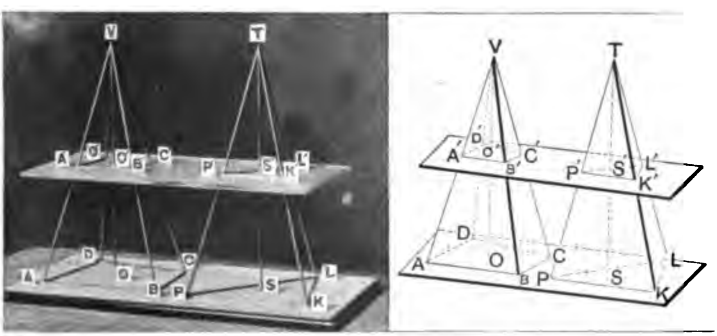
Hence $$\frac{RS}{AB} = \left(\frac{OS}{OB}\right) = \frac{ST}{BC} = \text{etc.}$$

Therefore $RSTUV$ is similar to $ABCDE$. Q. E. D.

661. COR. I. *The areas of any sections of a pyramid parallel to its base are proportional to the squares of their distances from the vertex.*

OUTLINE PROOF: $$\frac{\text{area } RSTUV}{\text{area } ABCDE} = \frac{\overline{RS}^2}{\overline{AB}^2} = \frac{\overline{OS}^2}{\overline{OB}^2} = \frac{\overline{OZ}^2}{\overline{OF}^2}. \qquad \S\ 380$$

662. COR. II. *If two pyramids having equal altitudes are cut by planes parallel to their bases at equal distances from their vertices, the sections thus formed will be proportional to the bases.*



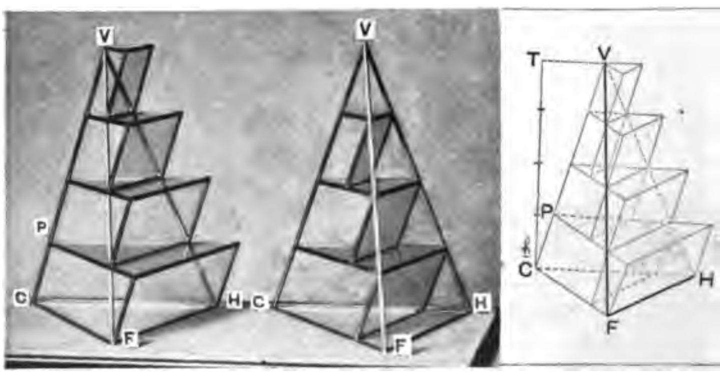
OUTLINE PROOF: $$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{VO'}^2}{\overline{VO}^2} = \frac{\overline{TS'}^2}{\overline{TS}^2} = \frac{\text{area } P'K'L'}{\text{area } PKL}$$ § 661

663. COR. III. *If two pyramids have equal altitudes and equivalent bases, sections parallel to their bases and equally distant from their vertices are equivalent.*

664. *Defs.*—A **triangular pyramid** is a pyramid whose base is a triangle; a **quadrangular pyramid**, one whose base is a quadrilateral.

PROPOSITION XIX. THEOREM

665. *The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.*



GIVEN—the triangular pyramid V-CFH, its altitude being CT.

TO PROVE—that its volume is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms of equal altitude, when their number is indefinitely increased.

Divide the altitude CT into any number of equal parts and call one of these parts h.

Through the points of division pass planes parallel to the base, forming triangular sections. § 660 II

Upon the base CFH and upon the sections as *lower* bases construct prisms having their lateral edges parallel to VC and their altitudes equal to h.

This set of prisms may be said to be *circumscribed* about the pyramid.

Also with the sections as *upper* bases construct prisms having their lateral edges parallel to VC and their altitudes equal to h.

This set of prisms may be said to be *inscribed* in the pyramid.

The first circumscribed prism (beginning at the top) is equivalent to the first inscribed prism, the second circumscribed to the second inscribed, and so on until the last circumscribed remains. § 642

Hence the sum of the inscribed prisms differs from the sum of the circumscribed by the lower circumscribed prism $P\text{-}CFH$.

But the volume of pyramid is greater than the total volume of the inscribed prisms and less than the total volume of the circumscribed prisms. Ax. 10

Therefore the difference between the pyramid and either of these totals is less than the difference between the totals themselves, i. e., less than the lower circumscribed prism.

But the volume of this prism is equal to the product of its base and altitude, and since its altitude can be indefinitely diminished, while its base remains the same, its volume can be made as small as we please.

That is, the total volume of the inscribed prisms, or the total volume of the circumscribed prisms, can be made to differ from the pyramid by less than any assigned volume.

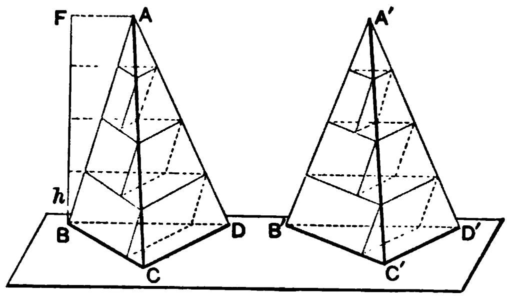
But they can never become equal to the pyramid. Ax. 10

Therefore the volume of the pyramid is their common limit. § 181

Q. E. D.

PROPOSITION XX. THEOREM

666. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



GIVEN—the triangular pyramids $A\text{-}BCD$ and $A'\text{-}B'C'D'$ having equivalent bases BCD and $B'C'D'$ in the same plane and having a common altitude BF.

TO PROVE the pyramids are equivalent.

Divide BF into any number of equal parts and denote one of these parts by h.

Through the points of division pass planes parallel to the bases and cutting the two pyramids.

The corresponding sections made by these planes in the two pyramids will be equivalent. § 663

Inscribe in each pyramid a series of prisms having the sections as upper bases and having the common altitude h.

The corresponding prisms, having equal altitudes and equivalent bases, will be equivalent. § 642

Therefore the total volume (or S) of the prisms inscribed in $A\text{-}BCD$ will equal the total volume (or S') of the prisms inscribed in $A'\text{-}B'C'D'$.

Suppose the number of divisions of the altitude BF to be indefinitely increased.

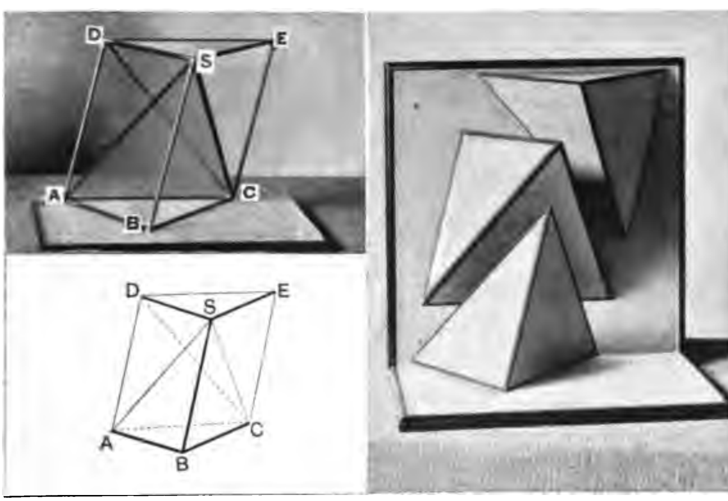
Then S will approach the volume of the pyramid $A\text{-}BCD$ as a limit, and S' will approach the volume of the pyramid $A'\text{-}B'C'D'$ as a limit. § 665

Since the variables S and S' are always equal to each other, their limits are equal. § 182

That is, the volumes of the pyramids are equal. Q. E. D.

PROPOSITION XXI. THEOREM

667. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.*

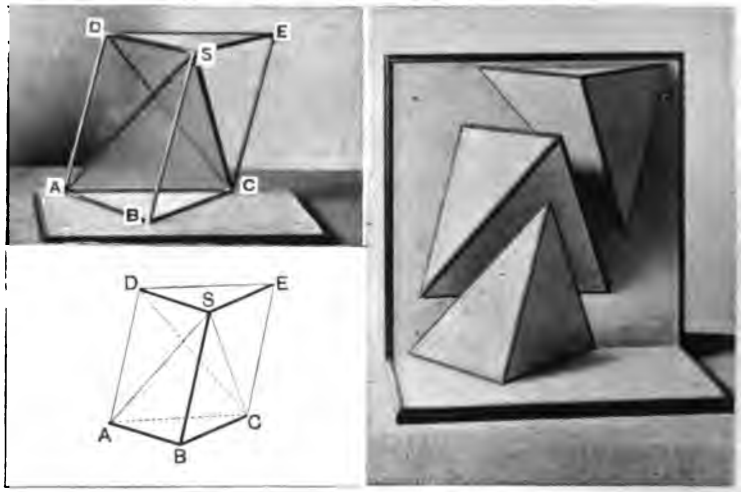


GIVEN the triangular pyramid $S\text{-}ABC$.

TO PROVE—its volume is one-third its base ABC by its altitude.

Construct a triangular prism having ABC as its base and BS as a lateral edge.

Taking away the triangular pyramid $S\text{-}ABC$ from the prism, the quadrangular pyramid $S\text{-}DACE$ remains.



Divide the latter by the plane *SDC* into two triangular pyramids *S-DAC* and *S-DEC*.

These pyramids have equal bases, the triangles *DCA* and *DCE*. § 114

They have equal altitudes, the perpendicular from the common vertex *S* upon the common plane of their bases.

Therefore they are equivalent. § 666

It can also be shown that the pyramids *S-ABC* and *S-DAC*, regarded as having the common vertex *C*, have equal bases and equal altitudes.

Hence these two pyramids are equivalent.

Hence all three are equivalent.

Therefore the pyramid *S-ABC* is one-third of the prism.

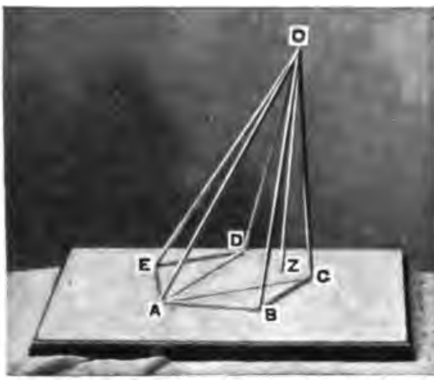
But the volume of the prism is equal to the product of its base and altitude. § 640

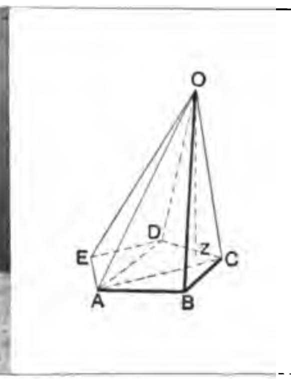
And the pyramid has the same base and altitude.

Hence the volume of the pyramid is equal to one-third the product of its base and altitude. Q. E. D.

PROPOSITION XXII. THEOREM

668. *The volume of any pyramid is equal to one-third the product of its base and altitude.*





GIVEN the pyramid O-$ABCDE$, whose altitude is OZ.

TO PROVE vol. O-$ABCDE = \frac{1}{3} ABCDE \times OZ$.

Divide $ABCDE$ into triangles by diagonals drawn from A.

Planes passed through OA and these diagonals will divide the pyramid into triangular pyramids, O-ABC, O-ACD, and O-ADE.

The volume of each triangular pyramid is one-third the product of its base and the common altitude OZ. § 667

Therefore the volume of the whole pyramid is equal to one-third the product of the sum of the bases of the triangular pyramids, i. e., the base of the whole pyramid and the common altitude. Q. E. D.

669. COR. I. *Pyramids having equivalent bases and equal altitudes are equivalent.*

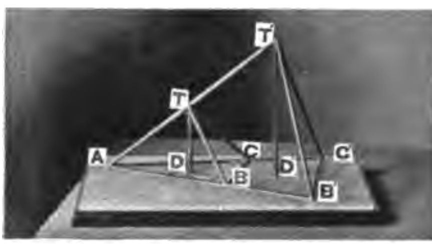
670. COR. II. *Any two pyramids are to each other as the products of their bases and altitudes.*

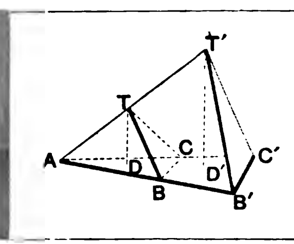
671. COR. III. *Two pyramids having equivalent bases are to each other as their altitudes.*

672. COR. IV. *Two pyramids having equal altitudes are to each other as their bases.*

PROPOSITION XXIII. THEOREM

673. *Two tetraedrons which have a triedral angle of the one equal to a triedral angle of the other are to each other as the products of the three edges about the equal triedral angles.*





GIVEN—the tetraedrons $TABC$ and $T'A'B'C'$ having the triedral angle A in common. Let V and V' denote their respective volumes.

TO PROVE $$\frac{V}{V'}=\frac{AB \times AC \times AT}{AB' \times AC' \times AT'}.$$

From T and T' let fall the perpendiculars TD and $T'D'$ upon the plane ABC.

$T'D'$ is in the plane ATD. § 557

Hence the three points A, D, and D' lie in one straight line.

Considering ABC and $AB'C'$ to be the bases of the tetraedrons,

$$\frac{V}{V'}=\frac{ABC \times TD}{AB'C' \times T'D'}=\frac{ABC}{AB'C'} \times \frac{TD}{T'D'}. \qquad \text{§ 670}$$

But $$\frac{ABC}{AB'C'}=\frac{AB \times AC}{AB' \times AC'}. \qquad \text{§ 377}$$

And since TD is parallel to $T'D'$, § 543

the triangles ATD and $A'T'D'$ are similar. § 262

Hence $$\frac{TD}{T'D'}=\frac{AT}{A'T'}.$$

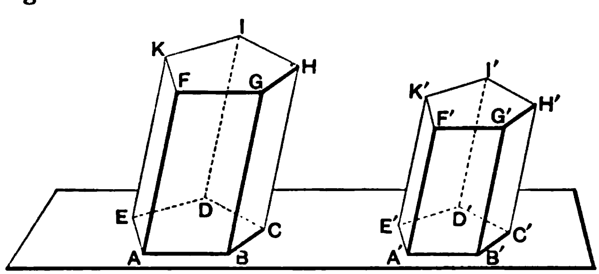
Therefore
$$\frac{V}{V'}=\frac{AB\times AC}{AB'\times AC'}\times\frac{AT}{AT'}=\frac{AB\times AC\times AT}{AB'\times AC'\times AT'}.$$ Q. E. D.

SIMILAR POLYEDRONS

674. *Def.*—Two polyedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and their homologous diedral angles are equal.

PROPOSITION XXIV. THEOREM

675. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



GIVEN—the similar polyedrons AH and $A'H'$ in which any two edges AB and CH of one are respectively homologous to $A'B'$ and $C'H'$ of the other.

TO PROVE $$\frac{AB}{A'B'}=\frac{CH}{C'H'}.$$

Since the faces $ABGF$ and $A'B'G'F'$ are similar,

$$\frac{AB}{A'B'} = \frac{BG}{B'G'}. \qquad \S\ 261$$

Since the faces $BCHG$ and $B'C'H'G'$ are similar,

$$\frac{CH}{C'H'} = \frac{BG}{B'G'}.$$

Therefore $$\frac{AB}{A'B'} = \frac{CH}{C'H'}. \qquad \text{Ax. 1}$$

Q. E. D.

676. *Def.*—The ratio of any two homologous edges of two similar polyedrons is the **ratio of similitude** of the polyedrons.

677. COR. I. *The ratio of any two homologous faces of two similar polyedrons is equal to the square of their ratio of similitude.*

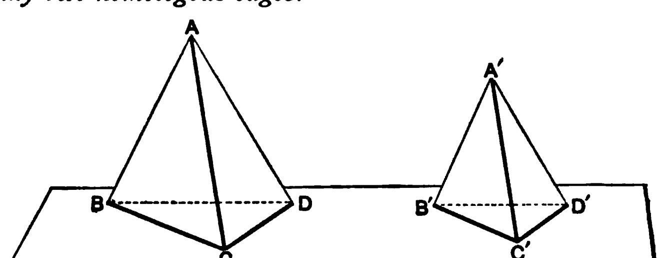
Hint.—Apply § 380.

678. COR. II. *The ratio of the total surfaces of two similar polyedrons is equal to the square of their ratio of similitude.*

Hint.—Apply § 251.

PROPOSITION XXV. THEOREM

679. *Two similar tetraedrons are to each other as the cubes of any two homologous edges.*



GIVEN the similar tetraedrons $ABCD$ and $A'B'C'D'$.

TO PROVE $$\frac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

The face angles of the triedral angles A and A' are equal each to each and similarly arranged.

Hence these triedral angles are equal. § 568

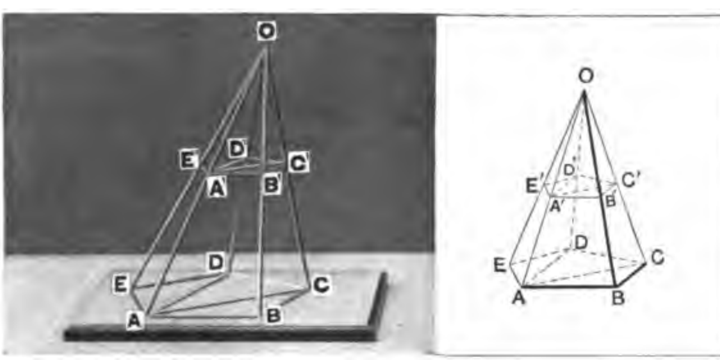
Therefore $\dfrac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \dfrac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}$ § 673

$= \dfrac{AB}{A'B'} \times \dfrac{AC}{A'C'} \times \dfrac{AD}{A'D'}$

$= \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'} \times \dfrac{AB}{A'B'}.$ § 675

That is, $\dfrac{\text{vol. } ABCD}{\text{vol. } A'B'C'D'} = \dfrac{\overline{AB}^3}{\overline{A'B'}^3}.$ Q. E. D.

680. COR. *If a pyramid is cut by a plane parallel to its base, the pyramid cut off is similar to the first, and the two pyramids are to each other as the cubes of any two homologous edges.*



Hint.— Prove the lateral edges divided proportionally and hence the homologous faces of the two pyramids similar; prove also the homologous diedral angles equal. The pyramids are therefore similar (§ 674). Divide the pyramids into similar triangular pyramids as shown in the figure.

Then, $\dfrac{\overline{OA}^3}{\overline{OA'}^3} = \dfrac{\text{vol. } O\text{-}ABC}{\text{vol. } O\text{-}A'B'C'} = \dfrac{\text{vol. } O\text{-}ACD}{\text{vol. } O\text{-}A'C'D'} = \dfrac{\text{vol. } O\text{-}ADE}{\text{vol. } O\text{-}A'D'E'}.$

Now apply § 251.

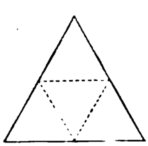
681. ***Remark.***—The ratio of the volumes of *any* two similar polyedrons is equal to the ratio of the cubes of any two homologous edges, that is, to the cube of the ratio of similitude. This is proved by means of exercises 697–703.

REGULAR POLYEDRONS

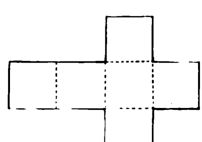
682. *Def.*—A **regular polyedron** is a polyedron whose faces are equal regular polygons, and whose diedral angles are all equal.

ICOSAEDRON DODECAEDRON OCTAEDRON HEXAEDRON TETRAEDRON

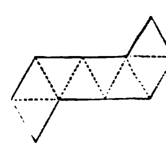
683. *Remark.*—There are but five regular polyedrons and these may be made from card-board as follows: Draw on card-board the figures given below, and on the inner lines cut the card-board half through with a penknife. Cut the figures out entire and fold the card-board as shown for the icosaedron in the accompanying plate.



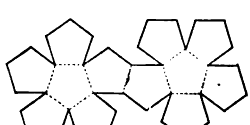
TETRAEDRON



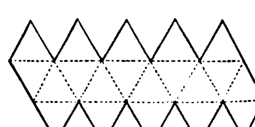
HEXAEDRON



OCTAEDRON



DODECAEDRON



ICOSAEDRON

ICOSAEDRON

PROBLEMS OF DEMONSTRATION

684. *Exercise.*—The four diagonals of a parallelopiped bisect each other.

685. *Exercise.*—Any straight line drawn through the intersection of the diagonals of a parallelopiped and terminated by two opposite faces is bisected in that point.

686. *Exercise.*—The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

687. *Exercise.*—In a rectangular parallelopiped, the four diagonals are equal to each other; and the square of a diagonal is equal to the sum of the squares of the three edges which meet at a common vertex.

688. *Exercise.*—In a quadrangular prism the two diagonals which connect either pair of opposite edges bisect each other.

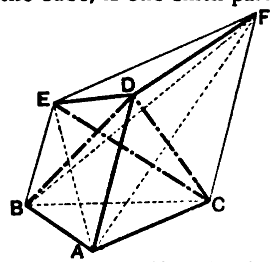
689. *Exercise.*—If a plane parallel to two opposite edges of a tetraedron cut the tetraedron, the section is a parallelogram.

690. *Exercise.*—If the angles at the vertex of a triangular pyramid are right angles, and the lateral edges are equal, prove that the sum of the perpendiculars on the lateral faces from any point in the base is constant.

691. *Exercise.*—The straight lines joining each vertex of a tetraedron with the intersection of the medians of the opposite face, meet in a point which divides each line into segments whose ratio is 3 : 1.

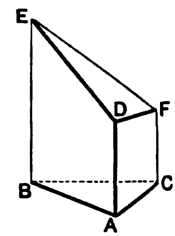
This point is called in Physics *the centre of gravity* of the tetraedron.

692. *Exercise.*—The pyramid whose base is one of the faces of a cube, and whose vertex is at the centre of the cube, is one-sixth part of the cube.

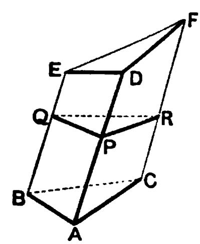


693. *Exercise.*—A truncated triangular prism is equivalent to the sum of three pyramids whose common base is either base of the truncated prism, and whose vertices are the three vertices of the other base.

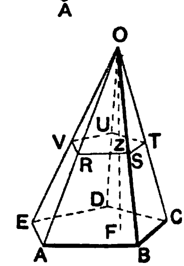
Hint.—Divide the truncated triangular prism into three triangular pyramids by the planes DBC and DEC. Show that the pyramids $D\text{-}BEC$ and $E\text{-}ABC$ are equivalent. Also the pyramids $D\text{-}CEF$ and $F\text{-}ABC$.



694. *Exercise.*—The volume of a right truncated triangular prism is equal to the product of one-third the sum of its lateral edges by the area of the base to which those edges are perpendicular.



695. *Exercise.*—The volume of any truncated triangular prism is equal to the product of one-third the sum of its lateral edges by the area of a right section.



696. *Exercise.*—A frustum of any pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

Hint.—To find the volume of the frustum, subtract the volume of the small pyramid from the volume of the large.

By the aid of § 380 express the result in terms of the bases and the altitude of the frustum.

697. *Exercise.*—Two polyedrons similar to a third are similar to each other.

Hint.—§ 674.

698. *Exercise.*—Two similar polyedrons are equal, if their ratio of similitude is unity.

Hint.—Show that the polyedrons can be made to coincide.

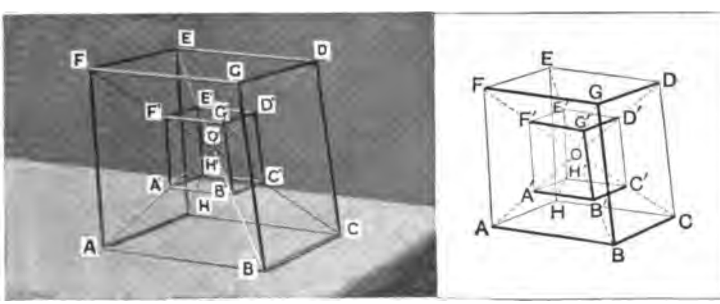
699. *Exercise.*—If two diedral angles have their faces respectively parallel and extending in the same direction, they are equal.

700. *Defs.*—If the vertices A, B, C, D, etc., of a polyedron are joined by straight lines to any point O, and the lines OA, OB, OC, OD, etc., are divided in the same ratio at the points A', B', C', D', etc., the polyedron $A'B'C'D'$, etc., is **radially situated** with regard to the polyedron $ABCD$, etc.

The ratio of the rays OA' and OA is the **ray ratio** of the two polyedrons.

The point O is the **ray centre.**

The point O is also called the **centre of similitude.**



701. *Exercise.*—Two radially situated polyedrons are similar, and their ratio of similitude is equal to the ray ratio.

702. *Exercise.*—Any two similar polyedrons can be radially placed, the ray ratio being equal to the ratio of similitude.

Hint.—Compare with § 289.

703. *Exercise.*—The ratio of the volumes of any two similar polyedrons is equal to the cube of their ratio of similitude.

Hint.—Let the polyedrons be radially placed, the smaller within the larger, and then apply § 680.

PROBLEMS OF CONSTRUCTION

704. *Exercise.*—Cut a cube by a plane so that the section shall be a regular hexagon.

705. *Exercise.*—Construct a parallelopiped of which three edges lie upon three given straight lines in space.

PROBLEMS FOR COMPUTATION

706. (1.) A rectangular block of marble is 1 m. 9 dcm. long, 9 dcm. 6 cm. broad, and 8 dcm. 9 cm. thick. What is its weight, if a cubic meter weighs 2675 kg.?

(2.) A barn with a gable roof is 60 ft. long, 30 ft. broad; the height from the floor to the eaves is 25 ft., to the gable $32\frac{1}{2}$ ft. Find its contents.

(3.) The area of the base of a right prism is 12 sq. in., its total area is 295 sq. in.; the base is a regular hexagon. What is the volume?

(4.) The great pyramid is estimated to have cost ten dollars a cubic yard, and three dollars besides for each square yard of surface; in this estimate the lateral faces are considered to be planes. The altitude of the pyramid is 488 ft., its base is 764 ft. square. What was its cost?

(5.) Express the volume of a cube in terms of the length of a diagonal.

(6.) The area of the lower base of a frustum of a pyramid is 100 sq. cm., of the upper base 30 sq. cm., and the altitude of the frustum is 5 dcm. What would be the altitude of the complete pyramid?

(7.) What is the volume of the frustum in (6)?

GEOMETRY OF SPACE

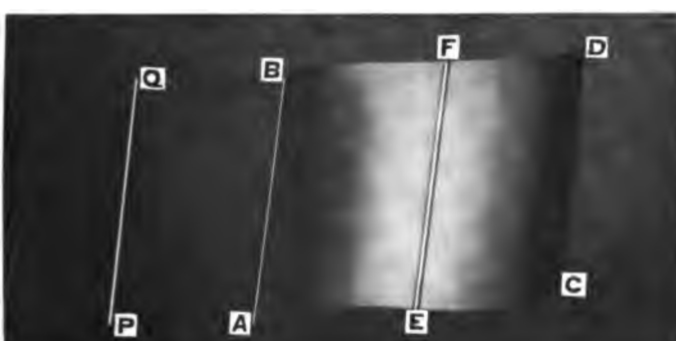
BOOK VIII

THE CYLINDER

707. *Def.*—A **curved line,** or **curve,** is a line no part of which is straight.

The curve may or may not lie entirely in one plane. An example of the first kind is the circumference of a circle; an example of the second kind is a curve like a corkscrew.

708. *Def.*—A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and is constantly parallel to a given fixed straight line.



Thus, if the straight line AB moves so as continually to intersect the curve AC and remains parallel to the line PQ, the surface generated, $ABDC$, is a cylindrical surface.

709. *Defs.*—The moving line is the **generatrix**; the fixed curve is the **directrix.**

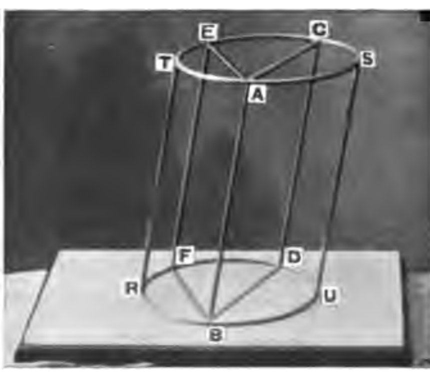
Any one position of the generatrix, as EF, is an **element** of the surface.

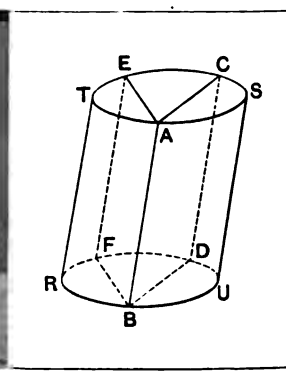
710. *Remark.*—The generatrix is usually supposed to be indefinite in extent, so that the surface generated is also of indefinite extent.

The directrix may be any curve whatever. But the proofs are rigorous only when the directrix is the circumference of a circle.

PROPOSITION I. THEOREM

711. *The sections of a closed cylindrical surface made by two parallel planes cutting the elements are equal.*





GIVEN—the closed cylindrical surface RS cut by two parallel planes, not parallel to the elements, in the sections TS and RU.

TO PROVE that TS and RU are equal.

Let A, C, and E be any three points in the perimeter of the upper section, and AB, CD, and EF the corresponding elements; B, D, and F being the points where these elements meet the perimeter of the lower section.

Pass planes through AB and CD, and through AB and EF.

Then AC is parallel to BD and AE to BF. § 524

Hence $AC=BD$ and $AE=BF$. § 116

The angles CAE and DBF are also equal. § 537

If, therefore, the planes of the two sections be superposed so that BD shall coincide with AC, F will fall on E.

If we suppose AC to be fixed and the point E to describe the perimeter of the upper section, then F will describe the perimeter of the lower section.

But in the superposed position of the sections F would always coincide with E.

Hence the perimeters of the two sections would coincide throughout. Therefore the sections are equal. Q. E. D.

712. *Defs.*—A **cylinder** is a solid bounded by a closed cylindrical surface and two parallel planes.

The cylindrical surface is the **lateral surface**, and the equal sections formed by the parallel planes are the **bases** of the cylinder.

CYLINDERS

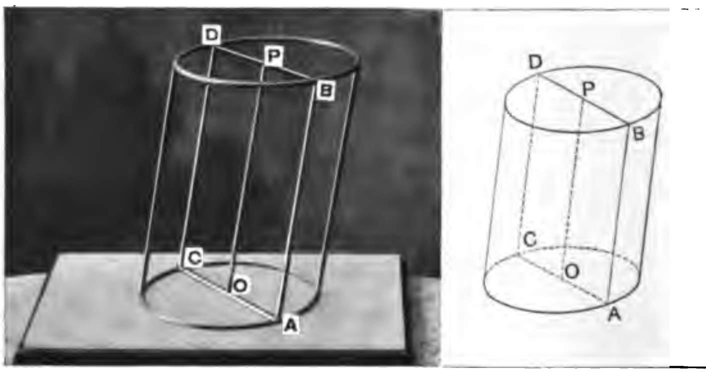
The term **element of a cylinder** is used to signify an element of its lateral surface.

713. *Def.*—A **right cylinder** is a cylinder whose elements are perpendicular to its bases.

714. *Defs.*—A **circular cylinder** is a cylinder whose bases are circles. The straight line joining the centres of its bases is the **axis** of the circular cylinder.

PROPOSITION II. THEOREM

715. *The axis of a circular cylinder is equal and parallel to its elements.*



GIVEN a circular cylinder AD, whose axis is OP.

TO PROVE—OP is equal and parallel to any element AB.

Draw through B and P the diameter BD of the upper base, and let CD be the element passing through D.

Then pass a plane through AB and CD cutting the lower base in AC.

AC is parallel to BD. § 524

Hence $AC = BD$. § 116

Therefore AC passes through O and is a diameter of the lower base. § 167

Hence $AO = BP$ § 153

Also AO was proved parallel to BP.

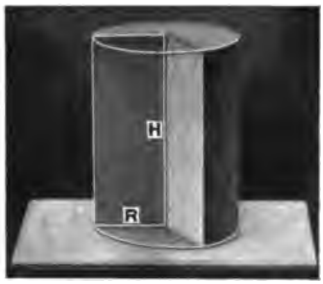
Hence the figure $ABPO$ is a parallelogram. § 123

Therefore OP is equal and parallel to AB. § 115

Q. E. D.

716. COR. I. *The axis of a circular cylinder passes through the centres of all sections parallel to its base.*

717. COR. II. *A right circular cylinder may be generated by the revolution of a rectangle about its side as an axis.*

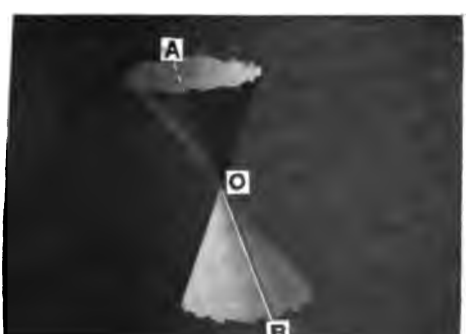


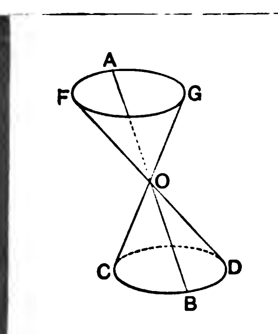
718. *Defs.*—For this reason a right circular cylinder is also called a **cylinder of revolution.**

The radius of the base of a cylinder of revolution is the **radius of the cylinder.**

THE CONE

719. *Def.*—A **conical surface** is a surface generated by a moving straight line which continually intersects a given fixed curve and constantly passes through a given fixed point.





Thus, if the straight line OB passes through the point O and moves so as continually to intersect the curve CD, the surface generated $O-CBD$ is a conical surface.

720. *Defs.*—The moving line is the **generatrix;** the fixed curve the **directrix;** the fixed point the **vertex.**

Any straight line in the surface, as OA, representing one position of the generatrix, is an **element** of the surface.

721. *Remark.*—If the generatrix is of indefinite length, as BOA, the conical surface consists of two symmetrical parts, each of indefinite extent, lying on opposite sides of the vertex, as $O-CBD$ and $O-GAF$.

The directrix may be any curve whatever. But the proofs are rigorous only when the directrix is the circumference of a circle.

722. *Defs.*—A **cone** is a solid bounded by a closed conical surface and a plane.

The conical surface is the **lateral surface** and the section made by the plane is the **base** of the cone.

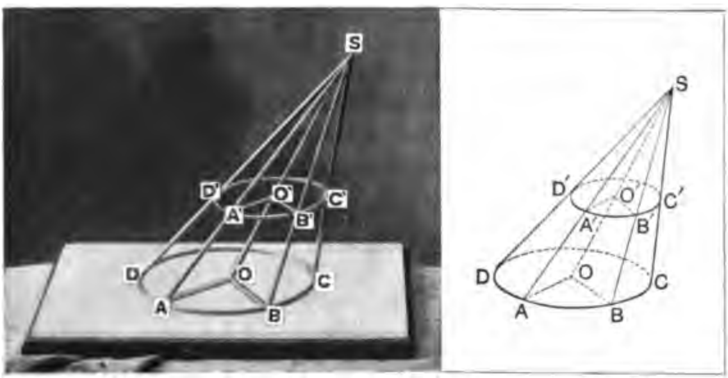
The vertex of the conical surface is the **vertex** of the cone, and the elements of the conical surface are also **elements of the cone.**

CONES

723. *Defs.*—A cone whose base is a circle is a **circular cone.** The straight line joining the vertex of a circular cone to the centre of its base is the **axis** of the cone.

PROPOSITION III. THEOREM

724. *Every section of a circular cone made by a plane parallel to its base is a circle, whose centre is the intersection of the axis with the plane parallel to the base.*



GIVEN—the circular cone $S—ABCD$ of which $A'B'C'D'$ is a section made by a plane parallel to its base.

Let the axis SO intersect the plane $A'B'C'D'$ in O'.

TO PROVE—that $A'B'C'D'$ is a circle and that O' is its centre.

Let A' and B' be any two points in the perimeter of $A'B'C'D'$.

Pass planes through SO and A', and through SO and B'.

Let SA and SB be the elements in which these planes intersect the conical surface, and AO, BO and $A'O', B'O'$ the straight lines in which they cut the parallel planes.

Then $A'O'$ is parallel to AO, and $B'O'$ to BO. § 524

Therefore the triangle SOA is similar to $SO'A'$, and SOB to $SO'B'$. § 262

Therefore $\frac{A'O'}{AO}=\frac{SO'}{SO}$ and $\frac{B'O'}{BO}=\frac{SO'}{SO}$. § 261

Hence $\dfrac{A'O'}{AO}=\dfrac{B'O'}{BO}.$

But $AO=BO.$ § 146

Therefore $A'O'=B'O'.$

Since A' and B' were taken as *any* two points in the perimeter of the section, all points in this perimeter are equidistant from O'.

Therefore $A'B'C'D'$ is a circle, and its centre is O'. Q. E. D.

725. *Def.*—A **right circular cone** is a circular cone whose axis is perpendicular to its base.

PROPOSITION IV. THEOREM

726. *A right circular cone may be generated by the revolution of a right triangle about one of its sides as an axis.*

The proof is left to the student.

727. *Def.*—From its mode of generation a right circular cone is also called a **cone of revolution.**

728. COR.—*The elements of a cone of revolution are all equal.*

THE SPHERE

729. *Defs.*—A **spherical surface** is a closed surface all points of which are equidistant from a point within, called the **centre.**

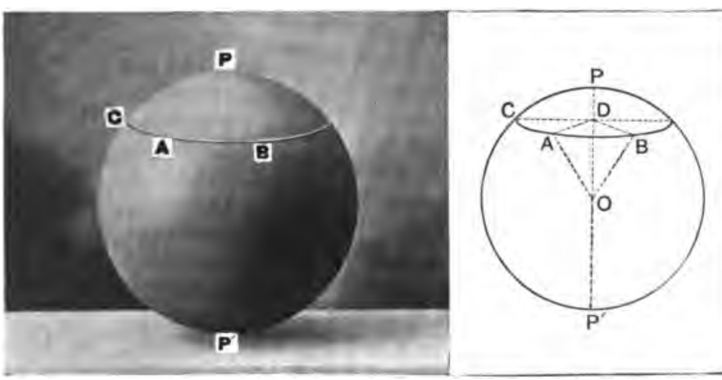
730. *Defs.*—A **sphere** is a solid bounded by a spherical surface.

A **radius** of the sphere is a straight line joining the centre to a point of the surface.

A **diameter** of the sphere is a straight line drawn through the centre and terminated at both ends by the surface.

PROPOSITION IV. THEOREM

731. *Every section of a sphere made by a plane is a circle whose centre is the foot of the perpendicular from the centre of the sphere on that plane.*



GIVEN—a sphere with centre O, cut by a plane in the section CAB.

Draw OD perpendicular to the cutting plane, meeting it at D.

TO PROVE—that CAB is a circle and that D is its centre.

Let A and B be any two points in the perimeter of CAB. Join AD, and BD.

$OA = OB$. § 729

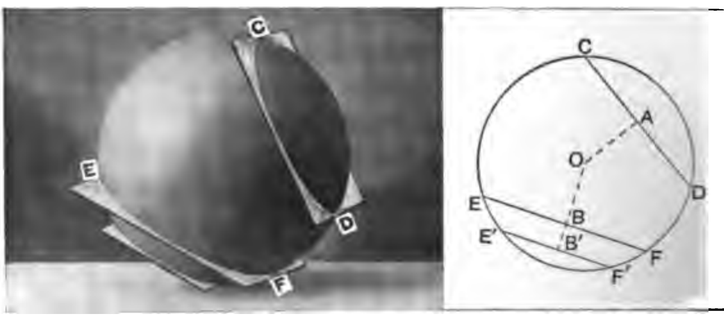
Therefore $DA = DB$. § 520 I

Since A and B are any two points in the perimeter of CAB, all points in this perimeter are equidistant from D.

Therefore CAB is a circle and D is its centre. Q. E. D.

732. COR. I. *If a plane is passed through the centre of a sphere, the centre of the circle thus formed is the centre of the sphere, and its radius is the radius of the sphere.*

733. COR. II. *Circles of the sphere equidistant from its centre are equal; and conversely.*



Hint.—Let fall perpendiculars OA and OB from the centre of the sphere on the planes of the two circles.

Then pass a plane through OA and OB intersecting the sphere in the circle $CDFE$ and the two circles in question in the diameters CD and EF.

The proof consists in applying §§ 167, 153.

734. COR. III. *The more distant a circle of the sphere is from the centre, the smaller is the circle; and conversely.*

735. *Def.*—A circle whose plane passes through the centre of the sphere is a **great circle.**

736. *Def.*—A circle whose plane does not pass through the centre of the sphere is a **small circle.**

737. COR. IV. *All great circles are equal.*

738. COR. V. *Any two great circles bisect each other.*

Hint.—Since they have the same centre, their intersection is a diameter of each circle.

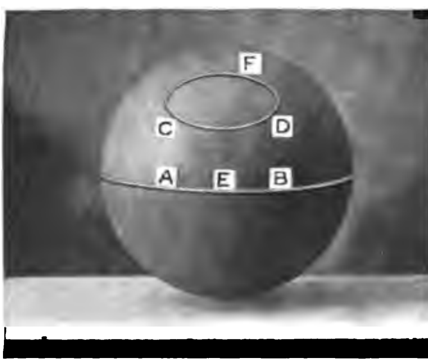
739. COR. VI. *Every great circle divides the sphere and its surface into two equal parts.*

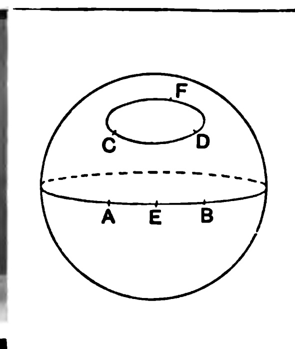
Hint.—Prove by superposition.

740. COR. VII. *Through any three points on the surface of a sphere one and only one circle can be drawn.*

741. COR. VIII. *Through any two points on the surface of a sphere, not at the extremities of a diameter, one and only one great circle can be drawn.*

Hint—The two points together with the centre of the sphere determine the plane of a great circle.





Question.—If the two points are at the extremities of a diameter, how is Corollary VIII. modified?

742. *Def.*—By the **distance** between two points on the surface of a sphere is usually meant the arc of a great circle, less than a semi-circumference, joining them.

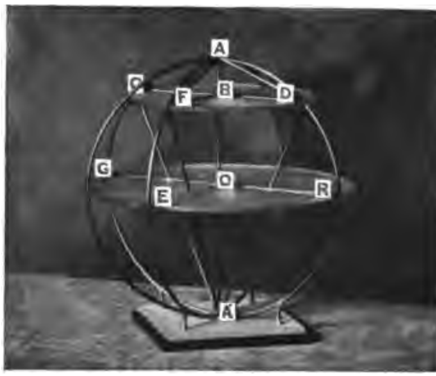
Thus the distance between the points A and B is the arc AEB.

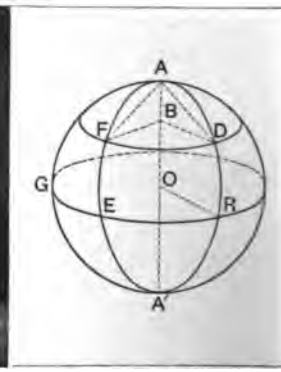
743. *Def.*—The diameter of a sphere which is perpendicular to the plane of a circle of the sphere is the **axis** of that circle.

744. *Def.*—The **poles** of a circle are the extremities of its axis.

PROPOSITION VI. THEOREM

745. *All points on the circumference of a circle of a sphere are equally distant from each of its poles.*





GIVEN—any two points F and D on the circumference of a circle CFD and A and A', the poles of CFD.

Draw the great-circle-arcs AF, AD, $A'F$, $A'D$.

TO PROVE arc AF=arc AD, and arc $A'F$=arc $A'D$.

Let B be the intersection of the axis AA' with the plane of CFD. Draw the straight lines AF and AD.

	$BF=BD$.	§ 731
Hence	chord AF=chord AD.	§ 519 I
Therefore	arc AF=arc AD.	§ 160
Similarly,	arc $A'F$=arc $A'D$.	Q. E. D.

746. *Def.*—The **polar distance** of a circle of a sphere is the arc of a great circle drawn from its nearer pole to any point of its circumference.

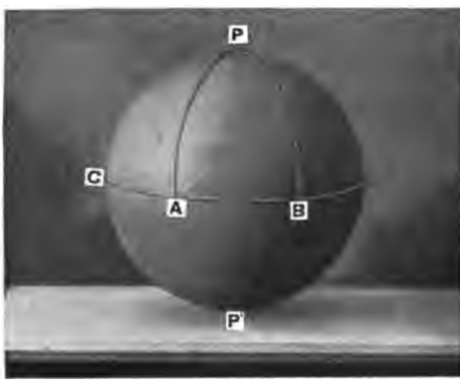
747. COR. *The polar distance of a great circle is a quadrant of a great circle.*

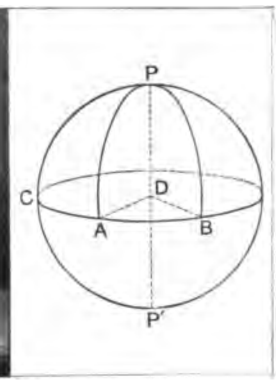
Hint.—Let GER be a great circle. Then its centre O is also the centre of the great circle ARA'. Hence the right angle AOR is measured by the arc AR.

748. *Def.*—The term **quadrant** in connection with a sphere is used to signify a quadrant of a great circle.

PROPOSITION VI. THEOREM

749. *If a point on the surface of a sphere is at a quadrant's distance from two points on that surface, it is the pole of the great circle passed through those points.*





GIVEN—a point P on the surface of a sphere at a quadrant's distance from each of the points A and B on that surface.

TO PROVE that P is the pole of the great circle AB.

Draw the radii DP, DA, and DB.

Since PA and PB are quadrants, PDA and PDB are right angles. §§ 732, 186

Therefore PD is perpendicular to the plane DAB. § 511

That is, P is the pole of the great circle AB. § 744

Q. E. D.

750. *Remark.*—The preceding theorems enable us to draw circumferences upon the surface of a sphere as easily as upon a plane. A pair of compasses with curved branches is employed. The opening of the compasses (distance between their points) is made equal to the chord of the polar distance of the required circle. Then, one point of the compasses being placed at the pole, the other describes the circumference.

SPHERICAL ANGLES

751. *Def.*—The **angle of two curves** meeting in a common point is the angle formed by the two tangents to the curves at that point.

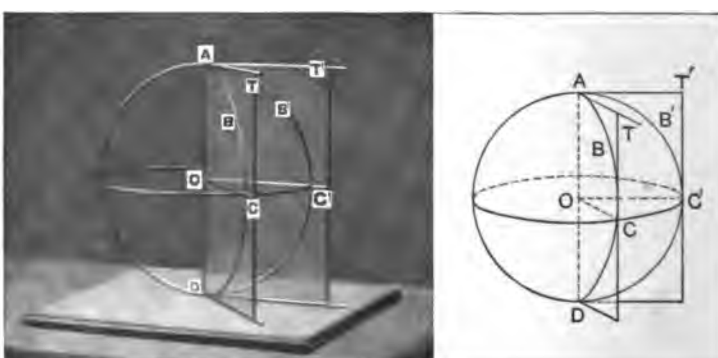
752. *Def.*—A **spherical angle** is the angle between two intersecting arcs of great circles on the surface of a sphere.

PROPOSITION VII. THEOREM

753. *The angle of two arcs of great circles on a spherical surface is*

I. *Equal to the plane angle of the diedral angle formed by their planes.*

II. *Measured by the arc of the great circle described with its vertex as a pole and included between its sides, produced if necessary.*



GIVEN—AB and AB', two arcs of great circles whose planes form a diedral angle having the diameter AD for edge.

With A as a pole describe a great circle cutting AB and AB', produced, if necessary, in C and C'.

I. TO PROVE—the angle BAB' is equal to the plane angle of the diedral angle $BADB'$.

Draw AT and AT' tangent to the arcs AB and AB' respectively.

Then by definition the angles BAB' and TAT' are identical. § 751

But AT and AT' are perpendicular to OA. § 170

Hence TAT', or BAB', is the plane angle of the diedral angle $BADB'$. § 547

Q. E. D.

II. TO PROVE—that the angle BAB' is measured by the arc CC'.

Join the centre of the sphere, O, to C and C'.

Then, since A is the pole of CC', the plane COC' is perpendicular to AO. § 744

Hence COC' is the plane angle of the diedral angle $BADB'$. § 510

Therefore the angle BAB' is equal to the angle COC'.

But COC' is measured by the arc CC'. § 183

Therefore BAB' is measured by the arc CC'. Q. E. D.

754. COR. I. *Any great-circle-arc AC, drawn through the pole of a given great circle CC' is perpendicular to the circumference CC'.*

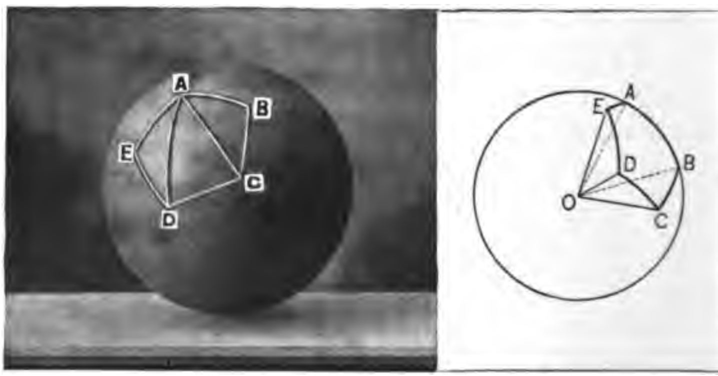
Hint.—First prove the plane AOC perpendicular to the plane COC'. Then the plane angle of the diedral angle OC is a right angle.

755. COR. II. Conversely, *any great-circle-arc perpendicular to a given arc must pass through the pole of the given arc.*

Hint.—Apply § 556.

SPHERICAL POLYGONS

756. *Defs.*—A **spherical polygon** is a portion of a spherical surface bounded by three or more arcs of great circles; as *ABCDE*.



The bounding arcs are the **sides** of the spherical polygon; their intersections, the **vertices**; the angles formed by the sides at the vertices, the **angles** of the spherical polygon.

757. *Def.*—A **diagonal** of a spherical polygon is an arc of a great circle joining any two vertices not consecutive.

758. *Remark.*—The sides of a spherical polygon are usually measured in degrees.

759. *Def.*—The polyedral angle, whose vertex is at the centre of the sphere, formed by the planes of the sides of a spherical polygon, is said to **correspond** to the spherical polygon.

Thus the polyedral angle $O-ABCDE$ corresponds to the spherical polygon $ABCDE$.

760. THEOREM. *The sides of a spherical polygon measure the corresponding face angles of the corresponding polyedral*

angle; and its angles are equal to the plane angles of the corresponding diedral angles.

Hint.—This proposition is an immediate consequence of §§ 183, 753 I.

761. *Remark.*—Since each face angle of a polyedral angle is assumed to be less than two right angles, each side of a spherical polygon will be assumed to be less than a semi-circumference.

762. *Def.*—The **parts** of a spherical polygon are its sides and angles.

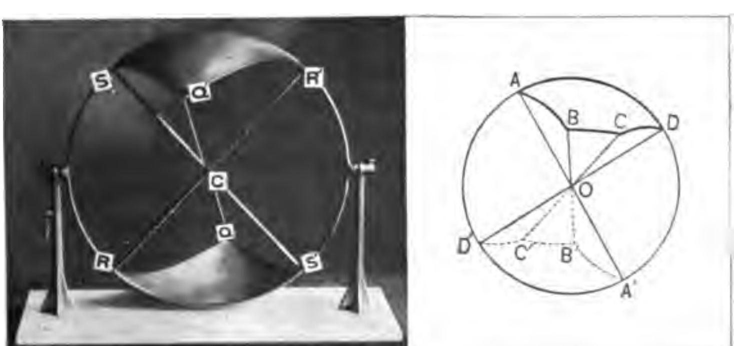
763. *Remark.*—By means of the relations between the parts of a spherical polygon and the parts of its corresponding polyedral angle we can infer from any property of polyedral angles, an analogous property of spherical polygons.

Reciprocally, from any property of spherical polygons, we can infer an analogous property of polyedral angles.

764. *Defs.*—A **spherical triangle** is a spherical polygon of three sides. It is **isosceles, equilateral,** or **right-angled** in the same cases in which a plane triangle would be.

SYMMETRICAL SPHERICAL TRIANGLES AND POLYGONS

765. *Def.*—Two spherical polygons are **vertical** when their vertices are situated by pairs at opposite ends of the same diameter.



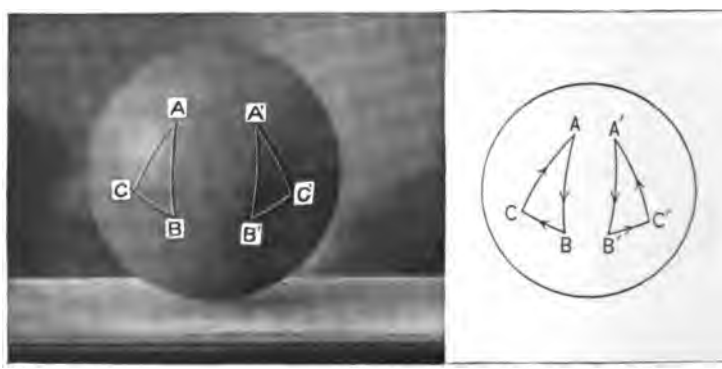
Thus, to determine the spherical polygon vertical to $ABCD$ draw the diameters AOA', BOB', COC', DOD'. Then $A'B'C'D'$ is vertical to $ABCD$.

766. THEOREM. *Two spherical polygons are vertical, if their corresponding polyedral angles are vertical, and conversely.*

Hint.—This follows immediately from the preceding definition and § 569.

767. *Def.*—Two spherical polygons are **symmetrical** when they have the same number of parts equal each to each and arranged in opposite order.

Thus, in the triangles ABC and $A'B'C'$, if $A=A'$, $B=B'$, $C=C'$, $AB=A'B'$, $BC=B'C'$, $CA=C'A'$, and the order of arrangement of the parts is opposite in the two figures, the triangles are symmetrical.



The meaning of the words "arranged in opposite order" will be made clearer by the following explanation:

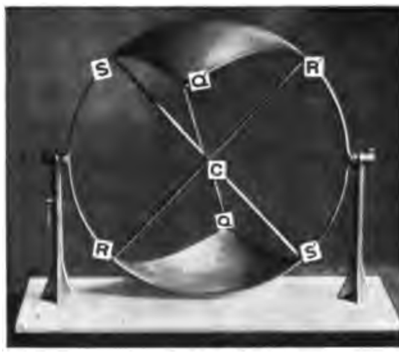
In the figure above the direction of motion in going from A to B to C to A is the direction of rotation of the hands of a clock, the direction of motion in going from A' to B' to C' to A' is opposite to the direction of rotation of the hands of a clock, if in each case we look at the surface of the sphere from the outside. If we look at the surface from the inside, the directions will be reversed.

768. THEOREM. *Two spherical polygons are symmetrical, if their corresponding polyedral angles are symmetrical, and conversely.*

This follows immediately from the preceding definition and §§ 570, 760.

PROPOSITION IX. THEOREM

769. *Two vertical spherical polygons are symmetrical.*



Proof.—The corresponding polyedral angles at the centre are vertical. § 766

They are therefore symmetrical. § 571

Hence the spherical polygons are symmetrical. § 768

Q. E. D.

PROPOSITION X. THEOREM

770. *Of two symmetrical spherical polygons either is equal to the vertical of the other.*

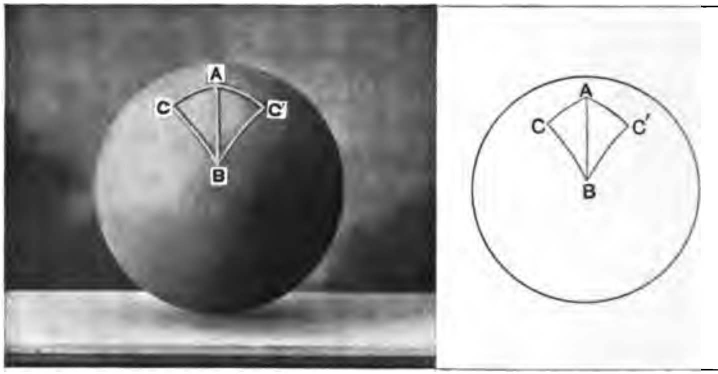
Proof.—The corresponding polyedral angles at the centre are symmetrical. § 768

Hence either may be made to coincide with the vertical of the other. § 572

When this is done, the two spherical polygons will be vertically opposite. § 766

Q. E. D.

771. *Remark.*—In general two symmetrical spherical polygons cannot be made to coincide, and hence are not equal.



Thus, if two symmetrical spherical triangles ABC and $A'B'C'$ are not isosceles, the only side of $A'B'C'$ with which AB can be made to coincide is $A'B'$. If we place A upon A' and B upon B', C and C' will fall on opposite sides of AB. If we place A upon B' and B upon A', C and C' will fall on the same side of AB, but will not coincide. But if the triangles are *isosceles*, they can be made to coincide, as the following proposition will show.

PROPOSITION X. THEOREM

772. *Two symmetrical isosceles spherical triangles are equal.*

Hint.—Place the angle A' in coincidence with its equal A. $A'B'$ will fall upon AC; but $A'B' = A'C' = AC$. Hence B' will fall upon C. Similarly C' will fall upon B. Show that the triangles coincide throughout.

773. COR. I. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*

Hint.—In superposing the symmetrical isosceles triangles in the above figure, the angle B' is made to coincide with C. But we know that $B' = B$.

774. COR. II. *If a spherical triangle is equilateral, it is also equiangular.*

775. COR. III. *If two face angles of a triedral angle are equal, the opposite diedral angles are equal.*

776. COR. IV. *If the three face angles of a triedral angle are equal, its three diedral angles are equal.*

PROPOSITION XII. THEOREM

777. *If two angles of a spherical triangle are equal, the opposite sides are equal.*

Hint.—Construct the symmetrical triangle. Prove that the two triangles are equal by the method employed in § 81.

778. COR. I. *If a spherical triangle is equiangular, it is also equilateral.*

779. COR. II. *If two diedral angles of a triedral angle are equal, the opposite face angles are equal.*

780. COR. III. *If the three diedral angles of a triedral angle are equal, the three face angles are equal.*

PROPOSITION XIII. THEOREM

781. *Any side of a spherical triangle is less than the sum of the two others.*

Hint.—Construct the corresponding triedral angle.
Then apply §§ 760, 565.

782. COR. I. *Any side of a spherical polygon is less than the sum of all the others.*

Hint.—Divide the polygon into triangles by diagonals from any vertex.

783. COR. II. *Any face angle of a polyedral angle is less than the sum of all the others.*

PROPOSITION XIV. THEOREM

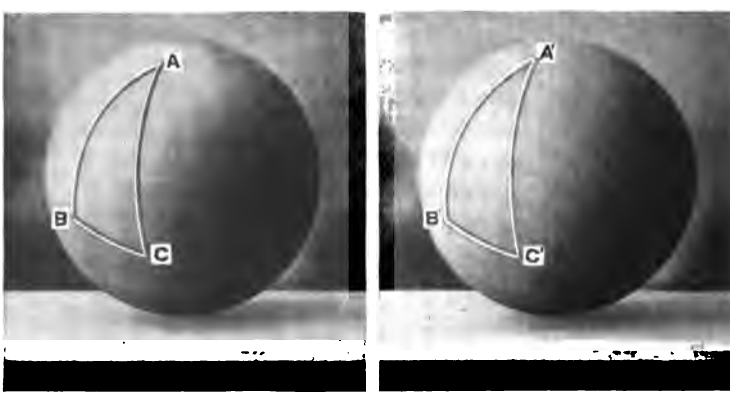
784. *Two triangles on the same sphere are equal:*

I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

III. *If the three sides of one are equal respectively to the three sides of the other.*

Provided in each case that the parts given equal are arranged in the same order in both triangles.



Proof—I, II. The proof for the corresponding propositions in Plane Geometry will apply. §§ 78, 81

III. The corresponding triedral angles are equal. § 568

Hence they can be placed in coincidence.

The two triangles will also coincide.

Therefore the triangles are equal. Q. E. D.

PROPOSITION XV. THEOREM

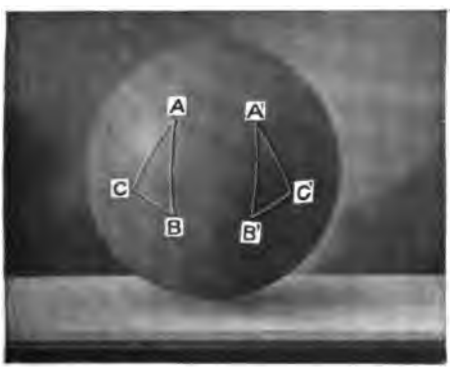
785. *Two triangles on the same sphere are symmetrical:*

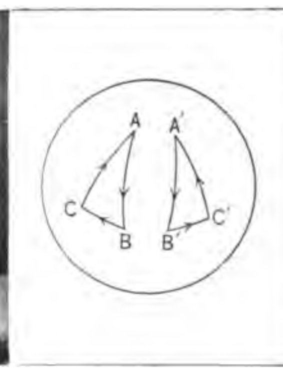
I. *If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

II. *If a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

III. *If the three sides of one are equal respectively to the three sides of the other.*

Provided in each case that the parts given equal are arranged in opposite order in the two triangles.





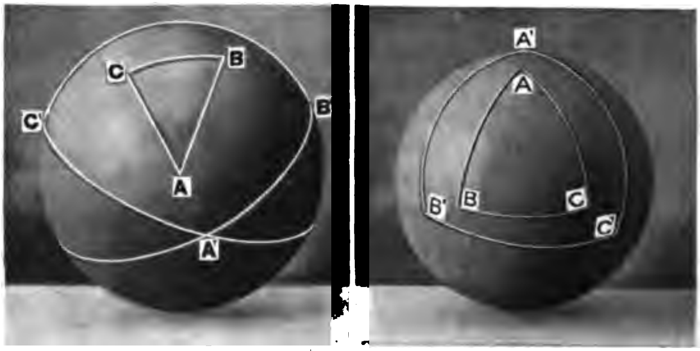
Proof.—Form the symmetrical triangle of the first triangle. This will be equal to the second triangle. § 784, I, II, III

Therefore the two given triangles are symmetrical.

Q. E. D.

POLAR TRIANGLES

786. *Def.*—If, with the vertices of a spherical triangle as poles, great circles are described, these circles will divide the spherical surface into eight triangles. One of these is the **polar triangle** of the given triangle.



The method of selecting the polar triangle from the eight is as follows: Let the given triangle be ABC and the polar triangle $A'B'C'$. Then A' is that one of the intersections of the arcs described from B and C as poles, which is less than a quadrant's distance from A. In a similar way B' and C' are determined. It is evident that the sides of a triangle and its polar triangle may intersect.

PROPOSITION XVI. THEOREM

787. *If one spherical triangle is the polar triangle of another, then, reciprocally, the second spherical triangle is the polar triangle of the first.*

GIVEN that $A'B'C'$ is the polar triangle of ABC.

TO PROVE that ABC is the polar triangle of $A'B'C'$.

Since B is the pole of $A'C'$, the distance $A'B$ is a quadrant; since C is the pole of $A'B'$, the distance $A'C$ is a quadrant. § 747

Therefore A' is the pole of BC. § 749

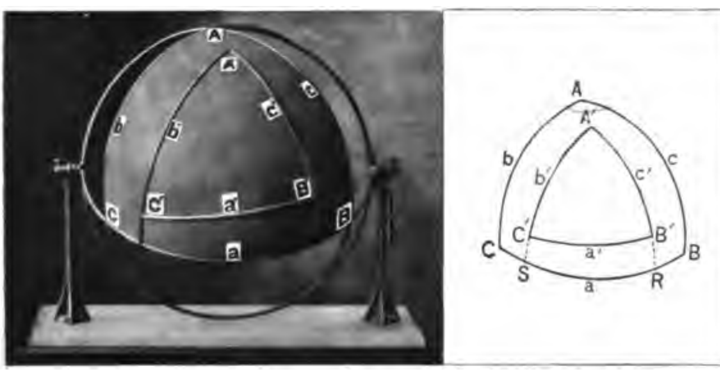
Similarly, B' is the pole of CA, and C' is the pole of AB.

Since also the distances AA', BB', and CC' are each less than a quadrant, ABC is the polar triangle of $A'B'C'$. § 786

Q. E. D.

PROPOSITION XVII. THEOREM

788. *In two polar triangles, each angle of one is measurea by the supplement of the side of which its vertex is the pole in the other.*



GIVEN — the polar triangles ABC and $A'B'C'$. Let A, B, C, and A', B', C' denote their angles, measured in degrees, and a, b, c, and a', b', c', the sides respectively opposite these angles, also measured in degrees.

TO PROVE—$A'+a=180°$, $B'+b=180°$, $C'+c=180°$,
$A+a'=180°$, $B+b'=180°$, $C+c'=180°$.

Produce $A'B'$ and $A'C'$ to meet BC at R and S.

Then, since B is the pole of $A'S$ and C the pole of $A'R$, BS and CR are quadrants. § 747

Therefore $BS+CR=180°$,

or $BR+RS+RS+SC=180°$,

or $RS+BC=180°$.

But $BC=a$, and RS measures the angle A'. § 753 II

Therefore $A'+a=180°$.

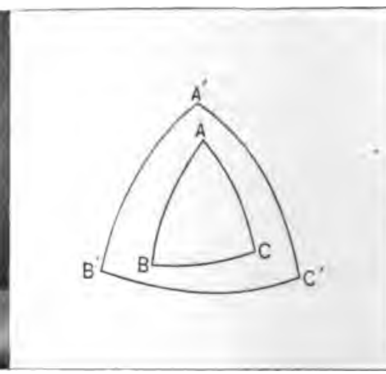
To prove that $A+a'=180°$ produce $B'C'$ to meet AB and AC.

In a similar manner the remaining relations are proved.

Q. E. D.

PROPOSITION XVIII. THEOREM

789. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



GIVEN the spherical triangle ABC.

Denote its angles by A, B, C, and the sides opposite in the polar triangle by a', b', c'.

TO PROVE $A+B+C>180°$ and $<540°$.

We have
$$A=180°-a'$$
$$B=180°-b'$$
$$C=180°-c'. \qquad \text{§ 788}$$

Adding these equations,
$$A+B+C=540°-(a'+b'+c').$$

Hence $A+B+C<540°$. Q. E. D.

Also since $a'+b'+c'<360°$, §§ 760, 567

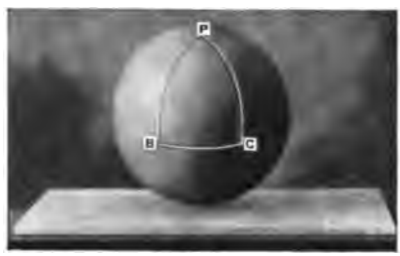
$A+B+C>180°$. Q. E. D.

790. COR. I. *A spherical triangle may have two, or even three, right angles; also two, or even three, obtuse angles.*

791. *Defs.*—A spherical triangle having two right angles is a **bi-rectangular triangle.**

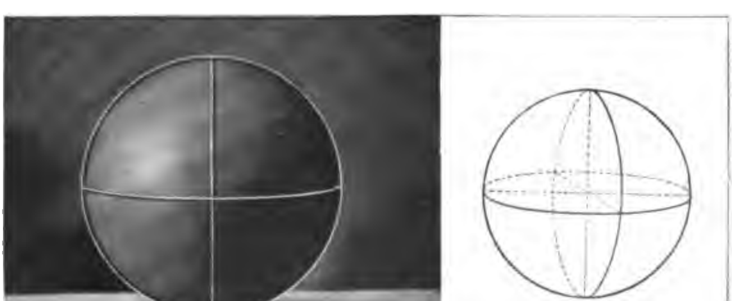
A spherical triangle having three right angles is a **tri-rectangular triangle.**

792. COR. II. *In a bi-rectangular triangle the sides opposite the right angles are quadrants.*



Hint.—Apply §§ 755, 747.

793. COR. III. *Three planes passed through the centre of a sphere, each perpendicular to the other two, divide the surface of the sphere into eight equal tri-rectangular triangles.*

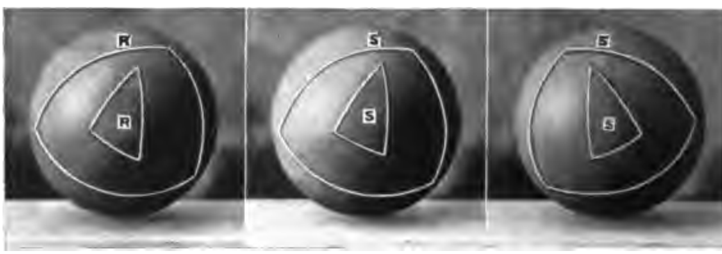


PROPOSITION XIX. THEOREM

794. *If two triangles on the same sphere are mutually equiangular:*

I. *They are equal, when the equal angles are arranged in the same order in both triangles.*

II. *They are symmetrical, when the equal angles are arranged in opposite order in the two triangles.*



GIVEN—two mutually equiangular spherical triangles R and S.

TO PROVE—that R and S are either equal or symmetrical.

Let R' and S' be the polar triangles of R and S respectively.

Since R and S are mutually equiangular, R' and S' are proved by Proposition XVII. to be mutually equilateral.

Hence R' and S' are either equal or symmetrical. §§ 784 III, 785 III

They are therefore mutually equiangular. § 767

Hence we can show that R and S, the polar triangles of R' and S', are mutually equilateral.

Therefore R and S are equal or symmetrical, according to the arrangement of their homologous parts. §§ 784 III, 785 III

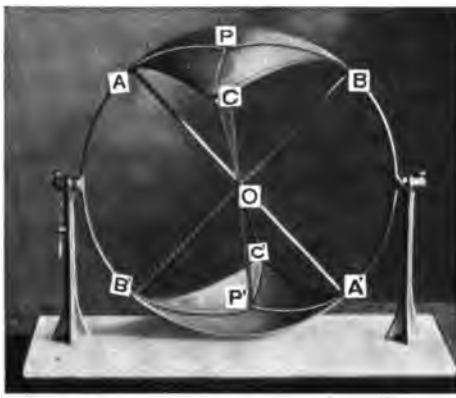
Q. E. D.

795. *Exercise.* — State the corresponding theorem for triedral angles.

MEASUREMENT OF SPHERICAL FIGURES

PROPOSITION XX. THEOREM

796. *Two symmetrical spherical triangles are equivalent.*



GIVEN two symmetrical triangles ABC and $A'B'C'$.

TO PROVE area ABC=area $A'B'C'$.

Let P be the pole of the small circle passing through A, B, and C, and draw the great-circle-arcs PA, PB, and PC.

Then $PA=PB=PC$. § 745

Now place the two triangles vertically opposite to each other and draw the diameter POP'. § 770

Also draw the great-circle-arcs $P'A'$, $P'B'$, and $P'C'$.

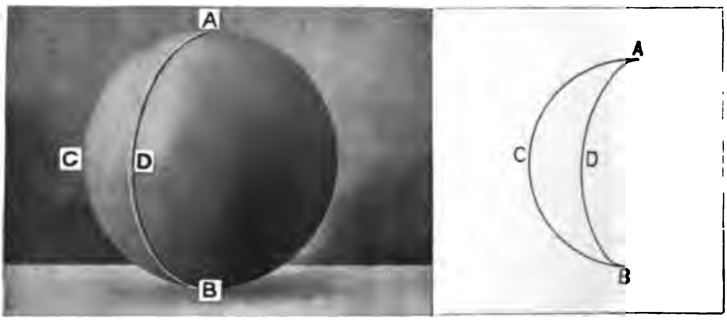
The vertical triangles PBC and $P'B'C'$ are symmetrical and isosceles and therefore equal. § 772

Similarly $PCA=P'C'A'$ and $PAB=P'A'B'$.

That is, the three parts of ABC are respectively equal to three parts of $A'B'C'$.

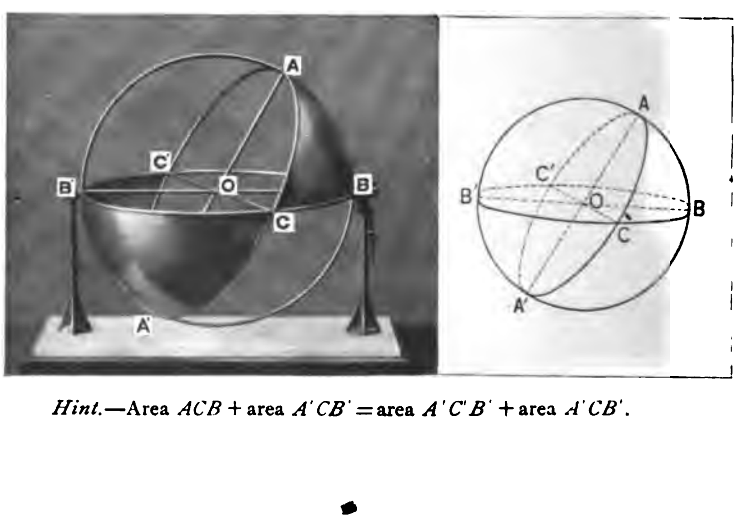
Therefore area ABC=area $A'B'C'$. Q. E. D.

797. *Defs.*—A **lune** is a portion of a spherical surface bounded by two semi-circumferences of great circles; as $ACBDA$.



The **angle of a lune** is the angle formed by its bounding arcs. Thus CAD is the angle of the lune $ACBDA$.

798. COR. I. *If two semi-circumferences of great circles BCB' and ACA' intersect on the surface of a hemisphere, the sum of the areas of the two opposite spherical triangles ACB and $A'CB'$ is equal to the area of a lune whose angle is equal to BCA.*

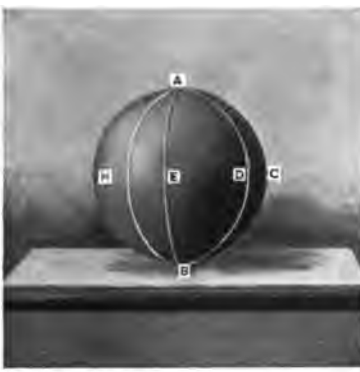


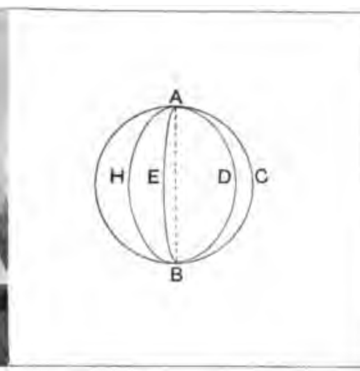
Hint.—Area ACB + area $A'CB'$ = area $A'C'B'$ + area $A'CB'$.

799. COR. II. *Two symmetrical spherical polygons are equivalent.*

PROPOSITION XXI. THEOREM

800. *Two lunes on the same sphere are equal, if their angles are equal.*





GIVEN—two lunes $ADBC$ and $AEBH$ on the same sphere, their angles DAC and HAE being equal.

TO PROVE that the lunes are equal.

Since the angles DAC and HAE are equal, the plane angles of the diedral angles $DABC$ and $HABE$ are equal. § 753 I

Hence these diedral angles are equal. § 552

They can therefore be superposed.

At the same time the lunes coincide.

Therefore the lunes are equal. Q. E. D.

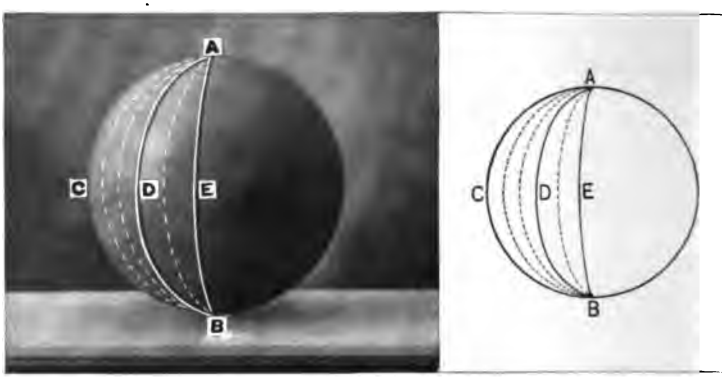
PROPOSITION XXII. THEOREM

801. *Two lunes on the same sphere are to each other as their angles.*

GIVEN—the lunes $ADBE$ and $ACBD$, whose angles are DAE and CAD.

TO PROVE $$\frac{ADBE}{ACBD}=\frac{DAE}{CAD}.$$

CASE I. *When the angles are commensurable.*



Suppose a common measure of DAE and CAD to be contained twice in DAE and three times in CAD.

Then $$\frac{DAE}{CAD}=\frac{2}{3}.$$ § 176

Draw from A to B semi-circumferences of great circles dividing the angles DAE and CAD into parts each equal to their common measure.

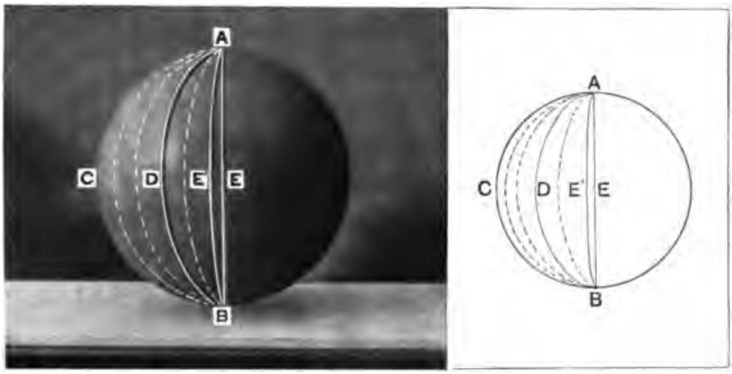
The small lunes thus formed are all equal. § 800

Of these lunes $ADBE$ contains two and $ACBD$ three.

Hence $$\frac{ADBE}{ACBD}=\frac{2}{3}.$$ § 176

Therefore $$\frac{ADBE}{ACBD}=\frac{DAE}{CAD}.$$ Q. E. D.

CASE II. *When the angles are incommensurable.*



Divide CAD into any number of equal parts by arcs of great circles drawn from A to B.

Apply one of these parts to DAE as many times as it will be contained in it, the final bounding arc taking the position $AE'B$.

Since the angles are incommensurable there will be a remainder $E'AE$ less than one of these parts.

The angles DAE' and CAD are commensurable.

Therefore $\frac{ADBE'}{ACBD}=\frac{DAE'}{CAD}$. Case I

Let the number of parts into which CAD is divided be indefinitely increased.

Then the angle DAE' will approach DAE as a limit. § 181

The lune $ADBE'$ will approach $ADBE$ as a limit.

Also $\frac{DAE'}{CAD}$ will approach $\frac{DAE}{CAD}$ as a limit.

And $\frac{ADBE'}{ACBD}$ will approach $\frac{ADBE}{ACBD}$ as a limit.

Therefore $\frac{ADBE}{ACBD}=\frac{DAE}{CAD}$. § 182

Q. E. D.

802. COR. I. *A lune is to the surface of the sphere on which it lies as the angle of the lune is to four right angles.*

Hint.—The surface of a sphere may be regarded as the limit of a lune whose angle approaches four right angles as a limit.

803. COR. II. Let A denote the angle of a lune measured in the right angle as a unit and L its surface measured in the tri-rectangular triangle as a unit.

Then the area of the spherical surface will be 8. § 793

Hence $\frac{L}{8}=\frac{A}{4}$. § 802

Therefore $L=2A$.

That is, *if the unit angle is the right angle and the unit surface the tri-rectangular triangle, a lune is measured by twice its angle.*

804. *Def.*—The **spherical excess** of a spherical triangle is the excess of the sum of its angles over two right angles.

If the angles are denoted by A, B, C, the spherical excess by E, and the right angle is the unit angle,

$$E=A+B+C-2.$$

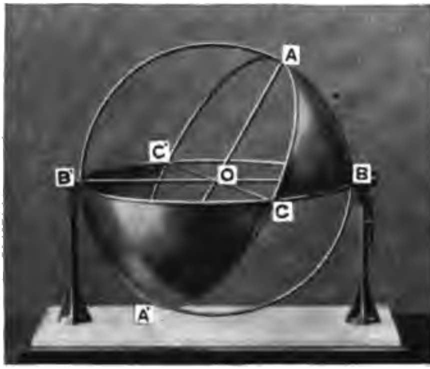
Thus, if the angles of a spherical triangle are 45°, 60°, 135°, its spherical excess is

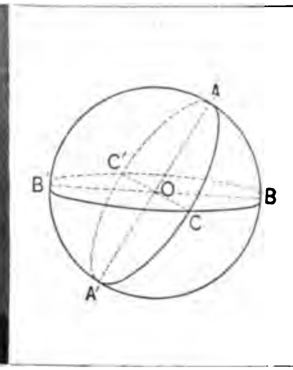
$$\left(\frac{45}{90}+\frac{60}{90}+\frac{135}{90}-2\right) \text{ right angles}=\frac{2}{3} \text{ right angle.}$$

805. *Exercise.*—If the angles of a spherical triangle are 50°, 100°, 130°, find its spherical excess.

PROPOSITION XXIII. THEOREM

806. *If the unit angle is the right angle and the unit surface the tri-rectangular triangle, the area of a spherical triangle is measured by its spherical excess.*





GIVEN the spherical triangle ABC.

TO PROVE area $ABC = A + B + C - 2$,

the unit angle being the right angle and the unit surface the surface of the tri-rectangular triangle.

Complete the circumference of which AB is an arc, and let BC and AC intersect it again in B' and A'.

Then, since BCA and $B'CA$ together form a lune whose angle is B,

area BCA + area $B'CA = 2B$. § 803

Similarly, area ACB + area $A'CB = 2A$.

Also the triangles ABC and $A'B'C$ are together equal to a lune whose angle is C. § 798

Hence area ABC + area $A'B'C = 2C$.

The sum of the areas of ABC, $B'CA$, $A'CB$, and $A'B'C$ is the area of the surface of a hemisphere, which with the adopted unit is 4.

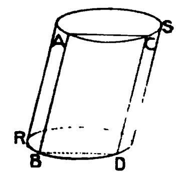
Hence, adding the three equations above,

2 area $ABC + 4 = 2A + 2B + 2C$.

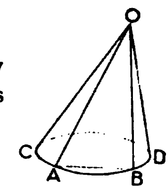
Therefore area $ABC = A + B + C - 2$. Q. E. D.

807. *Exercise.* — State a theorem for the area of *any* spherical polygon in terms of its angles.

PROBLEMS OF DEMONSTRATION



807. *Exercise.*—Every section of a cylinder made by a plane passing through an element is a parallelogram.



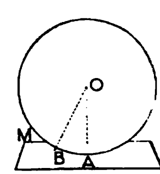
808. *Exercise.*—Every section of a cone made by a plane passing through its vertex and cutting its base is a triangle.

809. *Exercise.*—The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the straight line joining the centres of the two spherical surfaces, and whose centre is in that line.

810. *Exercise.*—If the centres of three spheres do not lie in the same straight line, their surfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centres and at equal distances from this plane on opposite sides.

811. *Defs.*—A plane or a line is **tangent** to a sphere when it has one, and only one, point in common with the surface of the sphere. This point is the **point of tangency.**

In the same case the sphere is tangent to the plane or line.



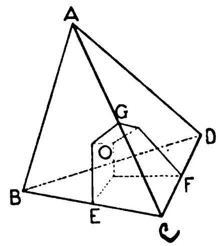
812. *Exercise.* — A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere; conversely, a plane tangent to a sphere is perpendicular to the radius drawn to the point of tangency.

813. *Exercise.*—If from a point without a sphere a tangent and a secant line be drawn, the square of the tangent is equal to the product of the whole secant and its external segment.

815. *Def.*—A sphere is **inscribed** in a polyedron when its centre is within the polyedron and its surface is tangent to all the faces of the polyedron.

816. *Exercise.*—A sphere can be inscribed in any tetraedron, and but one.

Hint.—The centre of the sphere is a point within the tetraedron equidistant from its four faces. Find such a point by means of § 559.



817. *Exercise.*—A spherical surface can be passed through any four points, not in the same plane, and but one.

Hint.—Determine the locus of points equidistant from A, B, and C, and from A, C, and D. The point common to these two loci will be the centre of the sphere.

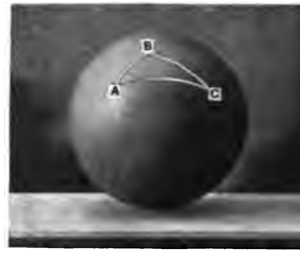
818. *Exercise.*—If any number of lines in space meet in a point, the feet of the perpendiculars drawn to these lines from another point lie on the surface of a sphere.

819. *Exercise.*—From a given point on the surface of a sphere, and not on a given great circle, but two great-circle-arcs can be drawn perpendicular to the given great circle; and these are the shortest and longest great-circle-arcs that can be drawn from the point to the given great circle.

820. *Exercise.*—If from a point within a spherical triangle arcs of great circles are drawn to the extremities of one side, the sum of these arcs is less than the sum of the two other sides of the triangle.

821. *Exercise.*—If two angles of a spherical triangle are unequal, the opposite sides are unequal, and the greater side is opposite the greater angle.

822. *Exercise.*—If two sides of a spherical triangle are unequal, the opposite angles are unequal, and the greater angle is opposite the greater side.

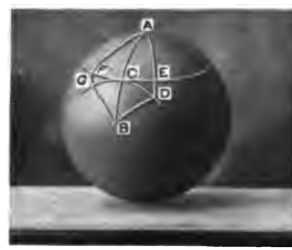


823. *Exercise.*—Any point in the bisector of a spherical angle is equally distant from the sides of the angle.

824. *Exercise.*—The bisectors of the angles of a spherical triangle meet in a point which is equally distant from the sides of the triangle.

825. *Exercise.*—The three medians of a spherical triangle meet in a point.

826. *Exercise.*—The perpendicular bisectors of the sides of a spherical triangle meet in a point.



827. *Exercise.*—The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle, not greater than a semicircumference, joining those points.

Hint.—If C be any point of AB, draw small circles with A and B as poles passing through C.

Prove that any line between A and B not containing C, as $AEDB$ is greater than some line passing through C.

828. *Exercise.*—If a, b, c are the sides of a spherical triangle and a', b', c' the corresponding sides of the polar triangle, if $a>b>c$, then $a'<b'<c'$.

829. *Exercise.*—Spherical triangles on equal spheres have equal areas if their polar triangles have equal perimeters.

LOCI

830. *Exercise.*—Find the locus of a point at a given distance from an indefinite straight line.

831. *Exercise.*—Find the locus of a point at a given distance from a straight line of definite length.

832. *Exercise.*—Find the locus of a point whose distance from a fixed straight line is in a given ratio to its distance from a fixed plane perpendicular to that line.

833. *Exercise.*—Find the locus of a point from which tangent lines drawn to three mutually intersecting spheres are equal.

834. *Exercise.*—Find the locus of the centre of a sphere which is tangent to three given planes.

835. *Exercise.*—Find the locus of the centre of the section of a given sphere made by a plane passing through a given point.

836. *Exercise.*—From a fixed point straight lines are drawn to the surface of a sphere. Find the locus of the points which divide these lines in a given ratio.

837. *Exercise.*—Find the locus of a point on the surface of a sphere equidistant from two given points on the surface.

838. *Exercise.*—Find the locus of a point on the surface of a sphere equidistant from three given points on the surface.

PROBLEMS OF CONSTRUCTION

839. *Exercise.*—Through a given straight line not intersecting a sphere pass a plane tangent to the sphere.

840. *Exercise.*—Construct a spherical surface of given radius:

(*a.*) Passing through three given points.

(*b.*) Passing through two given points and tangent to a given plane.

(*c.*) Passing through two given points and tangent to a given sphere.

(*d.*) Passing through a given point and tangent to two given planes.

(*e.*) Passing through a given point and tangent to two given spheres.

841. *Exercise.*—Bisect a given arc of a great circle.

842. *Exercise.*—Through a given point on a sphere draw a great circle tangent to a given small circle.

PROBLEMS FOR COMPUTATION

843. (1.) The radius of a sphere is 25 in. Find the area of a section made by a plane 10 in. distant from its centre.

(2.) What is the radius of a spherical surface passing through four points each of which is 9 cm. distant from the other three?

(3.) If the area of a spherical surface is 100 sq. ft., what is the area of a spherical triangle whose angles are $30°$, $120°$, and $150°$?

(4.) If the radius of a sphere is 10 in., what is the area of a spherical triangle whose angles are $70°$, $130°$, $100°$?

GEOMETRY OF SPACE

BOOK IX

MEASUREMENT OF THE CYLINDER, CONE, AND SPHERE

THE CYLINDER

844. *Def.*—A prism is **inscribed in a cylinder** when its lateral edges are elements of the cylinder and its bases are in the planes of the bases of the cylinder.

845. *Def.*—A plane is **tangent** to a cylinder when it passes through an element and meets the surface at no other point.

846. *Def.*—A prism is **circumscribed about a cylinder** when its lateral faces are tangent to the cylinder and its bases are in the planes of the bases of the cylinder.



847. *Def.*—A **right section** of a cylinder is a section made by a plane perpendicular to its elements.

848. *Remark.*—From the preceding definitions it follows that the bases of an inscribed prism are inscribed in the bases of the cylinder; the bases of a circumscribed prism are circumscribed about the bases of the cylinder; and a plane forming a right section of a cylinder forms a right section of each inscribed and circumscribed prism.

849. *Def.*—The **lateral area** of a cylinder is the area of its lateral surface.

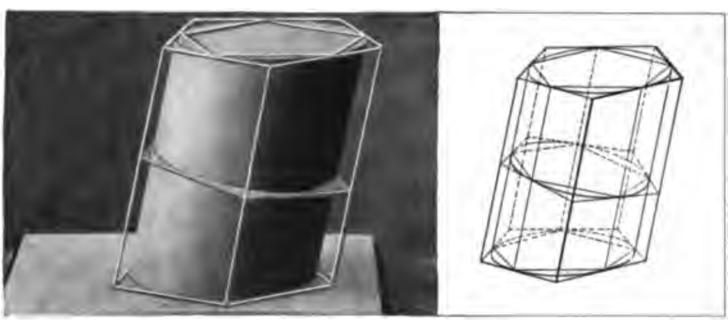
PROPOSITION I. THEOREM

850. *If the number of lateral faces of a prism inscribed in or circumscribed about a cylinder be indefinitely increased so that each one becomes indefinitely small, then*

I. *Any right section of the prism approaches a right section of the cylinder as a limit.*

II. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

III. *The volume of the prism approaches the volume of the cylinder as a limit.*



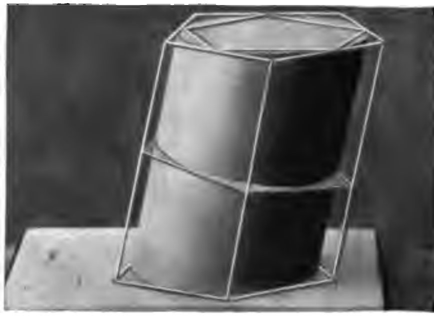
Proof.—I. A plane which forms a right section of the prism will also form a right section of the cylinder. § 848

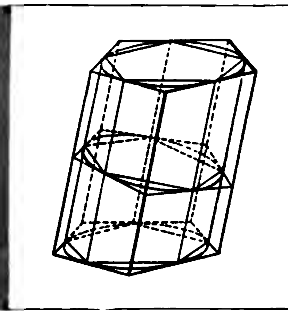
When the number of lateral faces of the prism is indefinitely increased so that each one becomes indefinitely small, the number of sides of the right section will be indefinitely increased, and each will become indefinitely small.

Therefore the right section of the prism approaches the right section of the cylinder as a limit. § 466

Q. E. D.

II. The lateral surface of the prism can be generated by a straight line moving about its right section as a directrix, provided this line remains parallel to the lateral edges and is terminated by the two bases. § 608





As the number of lateral faces increases indefinitely, the directrix of this line approaches the right section of the cylinder as a limit.

Hence the limit of the surface generated by this line is the surface generated by it when the directrix is the perimeter of the right section of the cylinder.

But this surface is the lateral surface of the cylinder. § 708

Therefore the limit of the lateral area of the prism is the lateral area of the cylinder. Q. E. D.

III. Let B', B'' be the respective bases of a circumscribed and corresponding inscribed prism, V', V'' their respective volumes, and H their common altitude.

Then $V' = B' \times H$, and $V'' = B'' \times H$. § 641

Hence $V' - V'' = (B' - B'') \times H$.

Now by increasing indefinitely the number of lateral faces of the prisms, and consequently the number of sides of their bases, the difference $B' - B''$ can be made as small as we please. § 466

Hence $(B' - B'') \times H$ can be made as small as we please.

Hence its equal $V' - V''$ can be made as small as we please.

But the volume of the cylinder is always less than V' and greater than V''. Ax. 10

Therefore the difference between the volume of the cylinder and either V' or V'' can be made as small as we please.

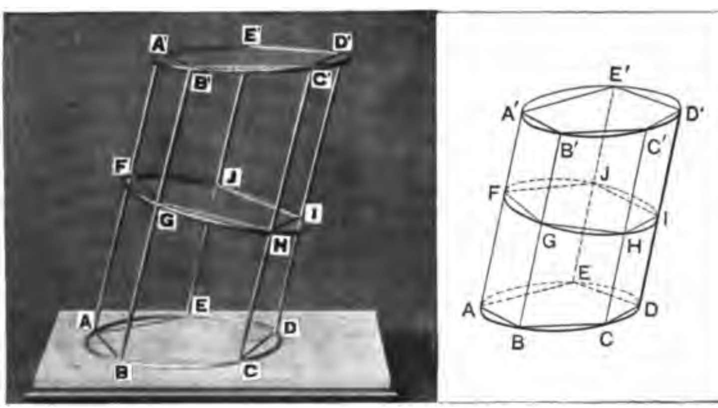
But V' and V'' can never equal the volume of the cylinder. Ax. 10

Therefore the volume of the cylinder is the common limit of V' and V''. § 181

Q. E. D.

PROPOSITION II. THEOREM

851. *The lateral area of a cylinder is equal to the product of the perimeter of a right section and an element.*



GIVEN—the cylinder AD', of which P is the perimeter of the right section $FGHIJ$, E an element, and S the lateral area.

TO PROVE $S = P \times E$.

Inscribe in the cylinder a prism. Let P' be the perimeter of its right section and S' its lateral area.

Its lateral edge is equal to E. § 844

Hence $S' = P' \times E$. § 614

Let the number of lateral faces of the prism be indefinitely increased.

Then S' approaches S as a limit, § 850 II

P' approaches P as a limit, § 850 I

and $P' \times E$ approaches $P \times E$ as a limit.

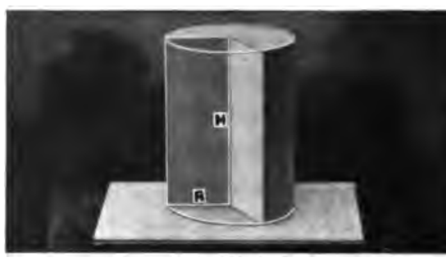
Therefore $S = P \times E$. § 182

Q. E. D.

852. *Def.*—The **altitude** of a cylinder is the perpendicular distance between its bases.

853. COR. I. *The lateral area of a right cylinder is equal to the product of the perimeter of its base and its altitude.*

854. COR. II. Let H denote the altitude, R the radius, S the lateral area, and T the total area of a cylinder of revolution.



Then $S = 2\pi RH$,

and $T = 2\pi RH + 2\pi R^2 = 2\pi R(H + R)$.

855. *Def.*—**Similar cylinders of revolution** are cylinders formed by the revolution of similar rectangles about homologous sides.

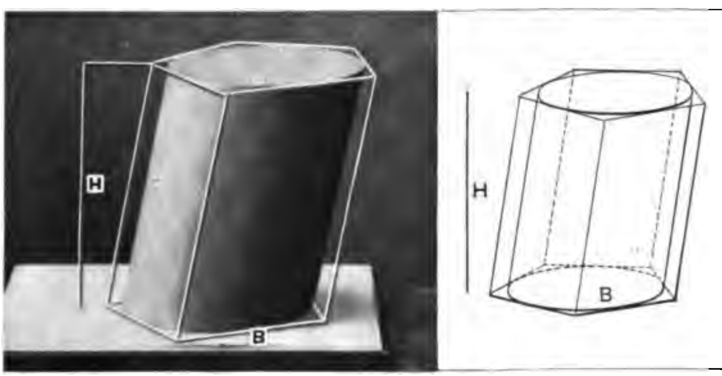
856. COR. III. *The lateral areas, or the total areas, of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of their radii.*

OUTLINE PROOF: $\frac{S}{s}=\frac{2\pi RH}{2\pi rh}=\frac{R}{r}\times\frac{H}{h}=\frac{H}{h}\times\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}.$

$$\frac{T}{t}=\frac{2\pi R(H+R)}{2\pi r(h+r)}=\frac{R}{r}\times\frac{H+R}{h+r}=\frac{H}{h}\times\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}.$$

PROPOSITION III. THEOREM

857. *The volume of a cylinder is equal to the product of its base and altitude.*



GIVEN—a cylinder, of which B is the base, H the altitude, and V the volume.

TO PROVE $V=B\times H.$

Circumscribe about the cylinder a prism. Denote its base by B' and its volume by V'.

Its altitude is H. § 545

Hence $V'=B'\times H.$ § 641

Let the number of lateral faces of the prism be indefinitely increased.

Then V' approaches V as a limit, § 850 III

B' approaches B as a limit, § 466

and $B' \times H$ approaches $B \times H$ as a limit.

Therefore $V = B \times H$. § 182

Q. E. D.

858. COR. I. Let H be the altitude, R the radius, and V the volume of a circular cylinder.

Then $V = \pi R^2 H$.

859. COR. II. *The volumes of two similar cylinders of revolution are to each other as the cubes of their altitudes, or as the cubes of their radii.*

OUTLINE PROOF : $\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2}{r^2} \times \frac{H}{h} = \frac{H^2}{h^2} \times \frac{H}{h} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

860. *Exercise.*—Find the volume generated by a rectangle 9 dcm. long and 4 dcm. broad (*a*) in revolving about its longer side; (*b*) in revolving about its shorter side.

861. *Exercise.*—Show that the volumes of two cylinders of revolution, whose lateral areas are equal, are to each other as their radii, or inversely as their altitudes.

THE CONE

862. *Def.*—A pyramid is **inscribed in a cone** when its lateral edges are elements of the cone and its base is in the plane of the base of the cone.

863. A plane is **tangent** to a cone when it passes through an element and meets the surface in no other point.

864. *Def.*—A pyramid is **circumscribed about a cone** when its lateral faces are tangent to the cone and its base is in the plane of the base of the cone.

865. *Remark.*—From these definitions it follows immediately that the base of an inscribed pyramid is inscribed in the base of the cone and that the base of a circumscribed pyramid is circumscribed about the base of the cone.

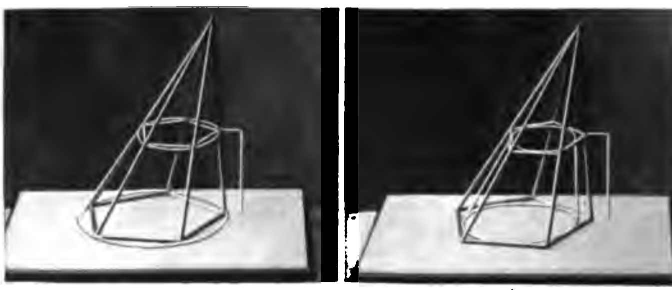
866. *Defs.*—A **truncated cone** is the portion of a cone contained between its base and a plane cutting all its elements.

The base of the cone and the section made by the cutting plane are the **bases** of the truncated cone.

FRUSTUM TRUNCATED CONE

867. *Def.*—A **frustum of a cone** is a truncated cone whose bases are parallel.

868. *Def.*—If a pyramid is inscribed in or circumscribed about a cone, a plane which cuts from the cone a frustum of cone cuts from the pyramid a frustum of a pyramid which is **inscribed** in or **circumscribed** about the frustum of a cone.



869. *Def.*—The **lateral area** of a cone is the area of its lateral surface.

PROPOSITION IV. THEOREM

870. *If the number of lateral faces of a pyramid inscribed in or circumscribed about a cone is indefinitely increased so that each one becomes indefinitely small, then*

I. *Any section of the pyramid approaches the section of the cone made by the same plane as a limit.*

II. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

III. *The volume of the pyramid approaches the volume of the cone as a limit.*

The proof of this proposition is analogous to that of Proposition I., and is therefore left to the student.

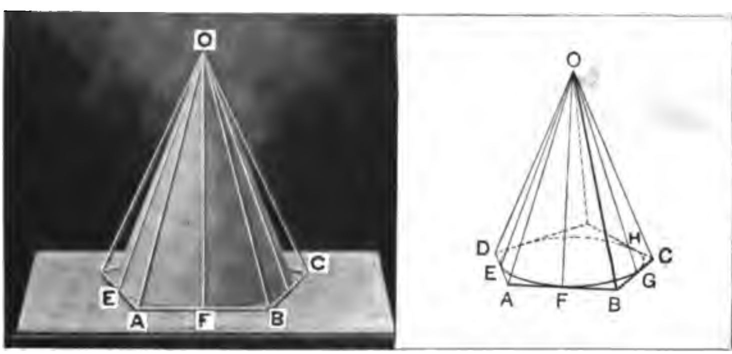
871. *Remark.*—The proposition obtained from the preceding by substituting the words "frustum of a pyramid" and "frustum of a cone" for "pyramid" and "cone" can be proved in the same way.

872. *Def.*—Any element of a cone of revolution is its **slant height.**

873. *Exercise.*—Prove that the slant height of a regular pyramid circumscribed about a cone of revolution is equal to the slant height of the cone of revolution.

PROPOSITION V. THEOREM

874. *The lateral area of a cone of revolution is equal to one-half the product of the circumference of its base and its slant height.*



GIVEN—the cone of revolution O-$EFGH$. Denote its slant height OE by L, the circumference of its base by C, and its lateral area by S.

TO PROVE $S=\frac{1}{2}C\times L$.

Circumscribe about the cone a regular pyramid. Denote the perimeter of its base by C' and its lateral area by S'.

Its slant height will also be L. § 873

Hence $S'=\frac{1}{2}C'\times L$. § 653

Let the number of lateral faces of the regular pyramid be indefinitely increased.

Then S' approaches S as a limit. § 870 II
And C' approaches C as a limit. § 466
Hence $\frac{1}{2}C' \times L$ approaches $\frac{1}{2}C \times L$ as a limit.
Therefore $S = \frac{1}{2}C \times L$. § 182

Q. E. D.

875. COR. I. Let R denote the radius, L the slant height, S the lateral area, and T the total area of a cone of revolution.

Then $S = \frac{1}{2} 2\pi R \times L = \pi RL.$
And $T = \pi RL + \pi R^2 = \pi R(L+R).$

876. COR. II. The formula for the lateral area may be written. $S = 2\pi \frac{R}{2} \times L.$

If K is the radius of a section half-way between the vertex and base, $K = \frac{1}{2}R.$
Therefore $S = 2\pi K \times L.$

That is, *the lateral area of a cone of revolution is equal to the product of the circumference of a section half-way between its vertex and base and its slant height.*

877. *Def.*—The **altitude** of a cone is the perpendicular distance from its vertex to its base.

878. *Exercise.*—What is the lateral area of a cone of revolution of which the altitude is 40 ft. and the radius of the base 30 ft.?

879. *Def.*—**Similar cones of revolution** are cones formed by the revolution of similar right triangles about homologous sides.

880. COR. III. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases.*

Hint.—The method of proof is the same as that followed in § 856.

881. *Def.*—The portion of an element of a cone of revolution included between the bases of a frustum is the **slant height** of the frustum.

882. *Exercise.*—Prove that the slant height of a frustum of a regular pyramid which is circumscribed about a frustum of a cone of revolution is equal to the slant height of the frustum of a cone.

PROPOSITION VI. THEOREM

883. *The lateral area of a frustum of a cone of revolution is equal to one-half the product of the sum of the circumferences of its bases and its slant height.*

GIVEN a frustum of a cone of revolution.

Denote the circumferences of its bases by C and c, its slant height by L, and its lateral area by S.

TO PROVE $S=\frac{1}{2}(C+c)\times L.$

Circumscribe about the frustum a frustum of a regular pyramid.

Denote the perimeters of its bases by C' and c', and its lateral area by S'. Its slant height will be L. § 882

Hence $S'=\frac{1}{2}(C'+c')\times L.$ § 658

Let the number of lateral faces of the frustum of a regular pyramid be indefinitely increased.

Then S' approaches S as a limit, § 871

$C'+c'$ approaches $C+c$ as a limit, § 466

and $\frac{1}{2}(C'+c')\times L$ approaches $\frac{1}{2}(C+c)\times L$ as a limit.

Therefore $S=\frac{1}{2}(C+c)\times L.$ § 182

Q. E. D.

884. COR. I. If R and r are the radii of the bases, L the slant height, and S the lateral area of a frustum of a cone of revolution,

$$S=\tfrac{1}{2}(2\pi R+2\pi r)\times L=\pi(R+r)\times L.$$

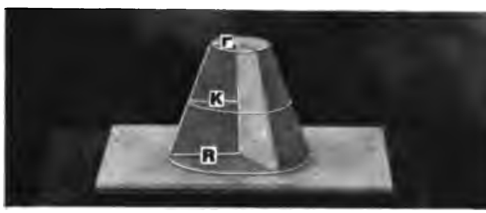
885. COR. II. The last formula may be written

$$S=2\pi\frac{R+r}{2}\times L.$$

If K is the radius of a section half-way between the bases of the frustum,

$$K=\frac{R+r}{2}.$$

Hence $$S=2\pi K\times L.$$



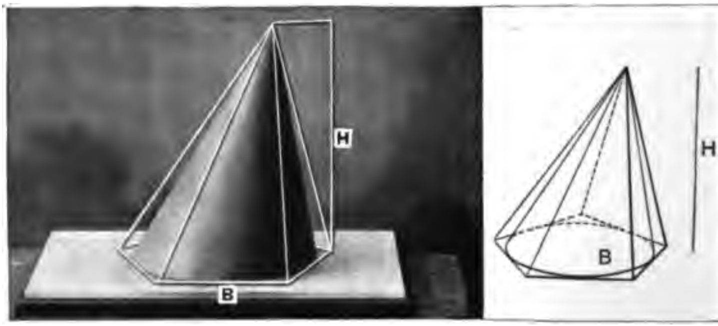
That is, *the lateral area of a frustum of a cone of revolution is equal to the product of the circumference of a section half-way between its bases and its slant height.*

886. *Exercise.*—Find the ratio of the lateral areas of a cylinder of revolution and a frustum of a cone of revolution, if the radii of the bases of the frustum are equal respectively to the radius and to one-half the radius of the cylinder, and the slant height of the frustum is equal to the altitude of the cylinder.

887. *Exercise.*—Find the lateral area of a frustum of a cone of revolution if the radii of the bases are 50 cm. and 30 cm. respectively, and the altitude of the cone from which the frustum is cut is 40 cm.

PROPOSITION VII. THEOREM

888. *The volume of a cone is equal to one-third the product of its base and altitude.*



GIVEN—a cone, of which B is the base, H the altitude, and V the volume.

TO PROVE $V = \frac{1}{3} B \times H$.

Circumscribe about the cone a pyramid. Denote its base by B', and its volume by V'. Its altitude is H.

Then $V' = \frac{1}{3} B' \times H$. § 668

Let the number of lateral faces of the pyramid be indefinitely increased.

Then V' approaches V as a limit, § 870 III

B' approaches B as a limit, § 466

and $\frac{1}{3} B' \times H$ approaches $\frac{1}{3} B \times H$ as a limit.

Therefore $V = \frac{1}{3} B \times H$. Q. E. D.

889. COR. I. If the base of the cone is a circle of radius R,

$$V = \frac{1}{3}\pi R^2 H.$$

890. COR. II. *The volumes of two similar cones of revolution are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*

Hint.—The method of proof is the same as that followed in § 859.

891. *Exercise.*—Find the volume of a cone of revolution of which the radius of the base is 12 in., the altitude 20 in.

THE SPHERE

892. *Defs.*—A **zone** is a portion of the surface of a sphere bounded by the circumferences of two circles whose planes are parallel.

The bounding circumferences are the **bases,** and the perpendicular distance between their planes is the **altitude** of the zone.

893. *Def.*—If the plane of one bounding circumference is tangent to the sphere, the zone is a **zone of one base.**

894. *Defs.*—A **spherical segment** is a portion of a sphere contained between two parallel planes.

The bounding circles are the **bases,** and the perpendicular distance between their planes is the **altitude** of the segment.

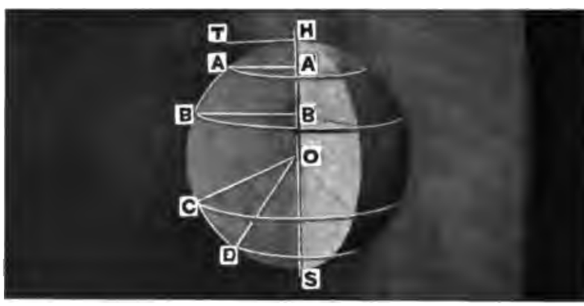
895. *Def.*—A **spherical segment of one base** is a spherical segment one of whose bounding planes is tangent to the sphere.

The curved surface of a spherical segment is a zone.

896. *Defs.*—If a semicircle is revolved about its diameter as an axis, the solid generated by any sector of the semicircle is a **spherical sector.**

The zone generated by the base of the sector of the semicircle is the **base** of the spherical sector.

897. *Remarks.*—Suppose a sphere generated by the revolution of the semicircle HAS about its diameter HS as an axis. Let AA' and BB' be two lines perpendicular to HS, and let OC and OD be radii of the semicircle.



Then the arc AB generates a zone whose altitude is $A'B'$; the points A and B generate the bases of the zone.

The arc HA generates a zone of one base.

The figure $AA'B'B$ generates a spherical segment whose altitude is $A'B'$; the lines AA' and BB' generate the two bases of the spherical segment.

The figure HAA' generates a spherical segment of one base whose altitude is HA'.

The sector COD of the semicircle generates a spherical sector. This spherical sector is bounded by three curved surfaces, namely, the two conical surfaces generated by the radii OC and OD, and the zone generated by the arc CD.

PROPOSITION VIII. LEMMA

898. *The area of the surface generated by a straight line revolving about an axis in its plane (not crossing the straight line) is equal to the product of the projection of the line on the axis and the circumference of the circle whose radius is the perpendicular to the line drawn at its middle point and terminated in the axis.*

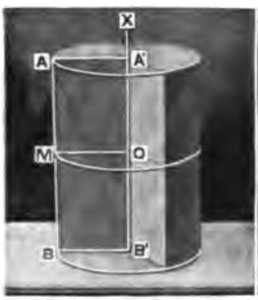


FIG. 1

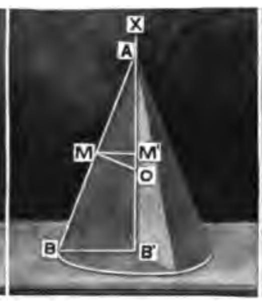


FIG. 2

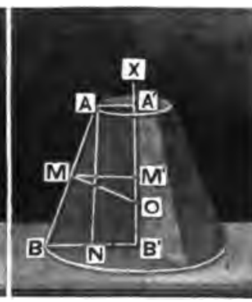


FIG. 3

GIVEN—the straight lines AB and XB' in the same plane, XB' not crossing AB. Let S denote the area of the surface generated by revolving AB about XB' as an axis.

Draw a perpendicular MO to AB at its middle point M cutting XB' in O, and let $A'B'$ be the projection of AB on XB'.

TO PROVE $S = A'B' \times 2\pi MO$.

CASE I. *When AB is parallel to XB'* (Fig. 1).

The surface generated in this case is the lateral surface of a cylinder of revolution. § 717

Hence $S = AB \times 2\pi BB'$. § 854

Or $S = A'B' \times 2\pi MO$. § 116

Q. E. D.

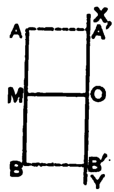


FIG. 1

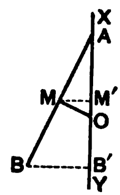


FIG. 2

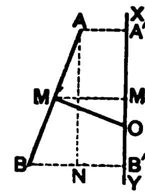


FIG. 3

CASE II. *When one end A of AB is in XB'* (Fig. 2).

The surface generated in this case is the lateral surface of a cone of revolution. § 726

Draw MM' perpendicular to XB'.

Then $S = AB \times 2\pi MM'$. § 876

The triangles $AB'B$ and $MM'O$ are similar. § 273

Hence $\dfrac{AB'}{AB} = \dfrac{MM'}{MO} = \dfrac{2\pi MM'}{2\pi MO}$. § 261

Hence $AB \times 2\pi MM' = AB' \times 2\pi MO$. § 237

Therefore $S = AB' \times 2\pi MO$. Q. E. D.

CASE III. *When AB is not parallel to XB' and does not meet XB'* (Fig. 3).

The surface generated in this case will be that of a frustum of a cone of revolution.

Draw MM' perpendicular to XB' and AN perpendicular to BB'.

Then $S = AB \times 2\pi MM'$. § 885

The triangles ANB and $MM'O$ are similar. § 273

Hence $\dfrac{AN}{AB} = \dfrac{MM'}{MO} = \dfrac{2\pi MM'}{2\pi MO}$.

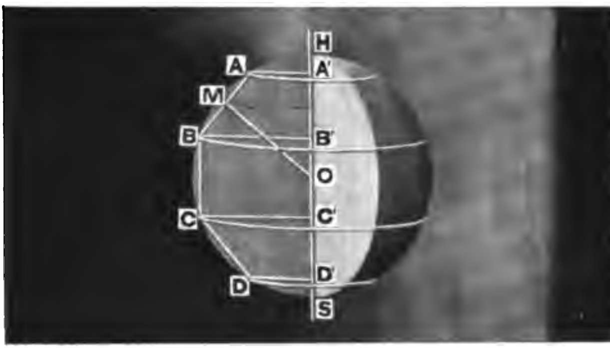
Hence $AB \times 2\pi MM' = AN \times 2\pi MO = A'B' \times 2\pi MO$.

Therefore $S = A'B' \times 2\pi MO$. Q. E. D.

899. *Def.*—A **broken line** is a line which is not straight, but consists of several straight parts.

PROPOSITION IX. THEOREM

900. *The area of a zone is equal to the product of its altitude and the circumference of a great circle.*



GIVEN—a zone formed by the revolution of the arc AD of the semicircle HAS about its diameter HS as an axis. Let $A'D'$ be the altitude of the zone and O the centre of the semicircle.

TO PROVE area zone $AD = A'D' \times 2\pi OA$.

Divide the arc AD into any number of equal parts, AB, BC, CD. Draw the chords AB, BC, CD.

Also draw AA', BB', CC', DD' perpendicular to HS and OM perpendicular to AB.

Denote by "area AB" the area of the surface generated by the straight line AB in revolving about HS.

Then area $AB = A'B' \times 2\pi OM$. § 898

Similarly area $BC = B'C' \times 2\pi OM$, §§ 160, 167

and area $CD = C'D' \times 2\pi OM$.

Adding these equations

$$\text{area broken line } ABCD = (A'B' + B'C' + C'D') \times 2\pi OM$$
$$= A'D' \times 2\pi OM.$$

Let the number of divisions of the arc AD be increased indefinitely.

Then the broken line approaches the arc AD as a limit and OM approaches the radius OA of the sphere as a limit.

Moreover, the limit of the surface generated by the broken line $ABCD$ is the surface generated by the limit of the broken line, that is, by the arc AD. This latter is the zone AD.

Therefore area zone $AD = A'D' \times 2\pi OA$. § 182

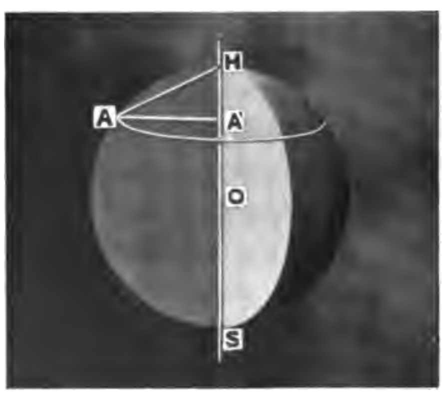
Q. E. D.

901. Cor. I. Let S denote the area of the zone, H its altitude, and R the radius of the sphere.

Then $$S = 2\pi RH.$$

902. Cor. II. *Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.*

903. Cor. III. *A zone of one base is equivalent to a circle whose radius is the chord of the generating arc of the zone.*



Outline proof: Area zone $HA = 2\pi OH \times HA' = \pi HS \times HA' = \pi \overline{HA}^2$.

904. Cor. IV. *The surface of a sphere is equivalent to four great circles.*

Hint.—The surface may be considered to be a zone whose altitude is the diameter of the sphere.

Hence its area is $$2\pi R \times 2R = 4\pi R^2.$$

905. COR. V. *Two spherical surfaces are to each other as the squares of their radii or as the squares of their diameters.*

PROPOSITION X. LEMMA

906. *The volume generated by a triangle revolving about an axis in its plane and passing through its vertex without crossing its surface, is equal to the product of one-third the altitude and the area generated by the base.*

GIVEN—the triangle ABC revolving about an axis XY passing through the vertex A without crossing the triangle. Let the altitude of the triangle be AD.

TO PROVE vol. gen. by $ABC =$ area $BC \times \frac{1}{3}AD$.

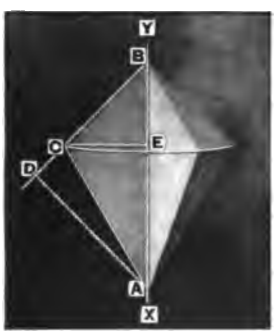


FIG. 1

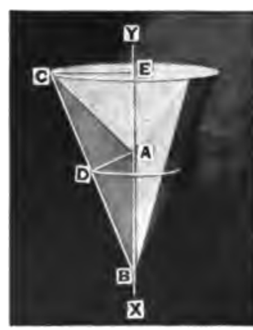


FIG. 2

CASE I. *When one side of the triangle ABC, as AB, lies in the axis.*

Draw CE perpendicular to the axis.

If this perpendicular falls within the triangle (Fig. 1), the volume generated by the triangle ABC is the sum of the volume generated by the triangles BEC and AEC. That is, vol. $ABC =$ vol. $BEC +$ vol. AEC. (1)

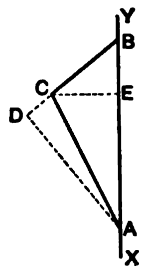


FIG. 1

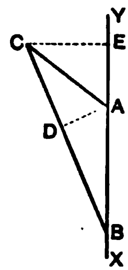


FIG. 2

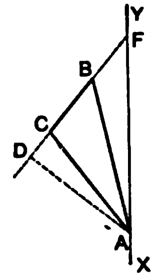


FIG. 3

If the perpendicular falls without the triangle (Fig. 2), the volume generated by the triangle ABC is the difference of the volumes generated by the triangles BEC and AEC. That is, vol. $ABC =$ vol. $BEC -$ vol. AEC. (2)

In either case

$$\text{vol. } BEC = \tfrac{1}{3}\pi \overline{EC}^2 \times BE \qquad \S\ 889$$

and

$$\text{vol. } AEC = \tfrac{1}{3}\pi \overline{EC}^2 \times AE.$$

Substituting these values in (1),

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times (BE + AE).$$

For this case $BE + AE = AB$.

Substituting in (2),

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times (BE - AE).$$

For this case $BE - AE = AB$.

Hence, in either case,

$$\text{vol. } ABC = \tfrac{1}{3}\pi \overline{EC}^2 \times AB$$

$$= \tfrac{1}{3}\pi EC \times EC \times AB.$$

But $EC \times AB = BC \times AD$,

since each is twice the area of the triangle ABC.

Therefore vol. $ABC = \tfrac{1}{3}\pi EC \times BC \times AD$.

But $\pi EC \times BC$ is the area of the conical surface generated by BC. § 875

Therefore vol. $ABC =$ area $BC \times \tfrac{1}{3}AD$. Q. E. D.

CASE II. *When neither side of the triangle ABC lies in the axis, and the base BC produced meets the axis in F* (Fig. 3).

Then $\quad$ vol. ABC = vol. AFC − vol. AFB.
But $\quad$ vol. AFC = area $FC \times \frac{1}{3}AD$,
and $\quad$ vol. AFB = area $FB \times \frac{1}{3}AD$. $\quad$ Case I
Therefore vol. ABC = (area FC − area FB) $\times \frac{1}{3}AD$
= area $BC \times \frac{1}{3}AD$. $\quad$ Q. E. D.

CASE III. *When the base BC of the triangle ABC is parallel to the axis.*

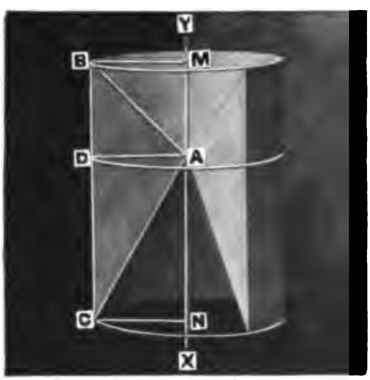


FIG. 4

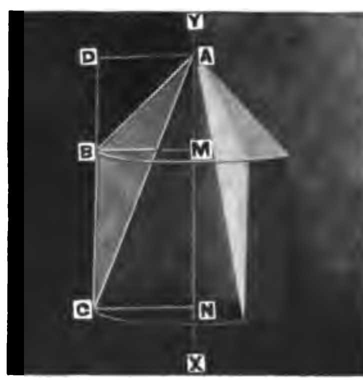


FIG. 5

According as AD falls within (Fig. 4) or without (Fig. 5) the triangle,

vol. ABC = vol. ADC + vol. ADB, $\quad$ (3)
or $\quad$ vol. ABC = vol. ADC − vol. ADB. $\quad$ (4)

Draw BM and CN perpendicular to XY.

For either figure

vol. ADC = vol. $ADCN$ − vol. ACN
$= \pi\overline{NC}^2 \times AN - \frac{1}{3}\pi\overline{NC}^2 \times AN$ $\quad$ §§ 858, 889

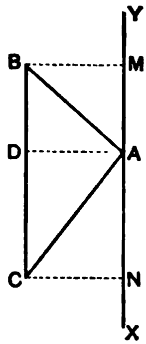


FIG. 4

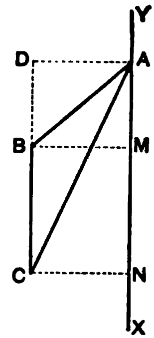


FIG. 5

$$=\tfrac{2}{3}\pi\overline{NC}^2\times AN=\tfrac{2}{3}\pi AD^2\times DC$$
$$=2\pi AD\times DC\times\tfrac{1}{3}AD.$$

But $2\pi AD\times DC$ is the area of the cylindrical surface generated by DC. § 854

Therefore vol. $ADC=$area $DC\times\frac{1}{3}AD$.

Similarly vol. $ADB=$area $DB\times\frac{1}{3}AD$.

Substituting these values for vol. ADC and vol. ADB in equations (3) and (4), and remembering that equation (3) applies to Fig. 4 and equation (4) to Fig. 5,

vol. $ABC=$area $BC\times\frac{1}{3}AD$. Q. E. D.

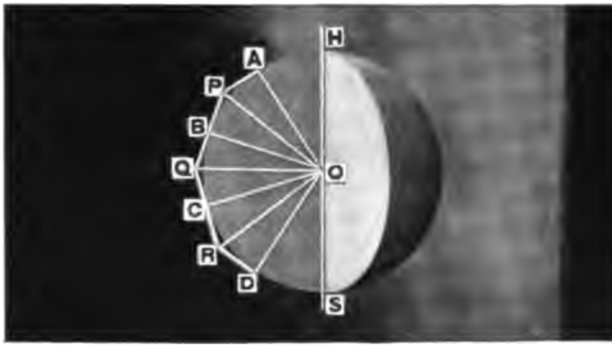
PROPOSITION XI. THEOREM

907. *The volume of a spherical sector is equal to the product of the area of the zone which forms its base and one-third the radius of the sphere.*

GIVEN—a spherical sector, formed by the revolution of the sector AOD of the semicircle HAS about its diameter HS as an axis.

TO PROVE—vol. sph. sect. $AOD=$area zone $AD\times\frac{1}{3}OA$.

Divide the arc AD into any number of equal parts, AB, BC, CD.



At A, B, C, and D draw tangents AP, PQ, QR, RD. Draw OB, OC, OP, OQ, OR.

The volume generated by the polygon $OAPQRD$ is the sum of the volumes generated by the triangles OAP, OPQ, OQR, ORD.

But vol. AOP = area $AP \times \frac{1}{3}OA$ § 906

vol. OPQ = area $PQ \times \frac{1}{3}OB$ = area $PQ \times \frac{1}{3}OA$

etc.

Hence

vol. $OAPQRD$ = (area AP + area PQ + etc.) $\times \frac{1}{3}OA$

= area $APQRD \times \frac{1}{3}OA$.

Let the number of divisions of the arc AD be indefinitely increased.

Then broken line $APQRD$ approaches arc AD as a limit;

surface generated by the broken line approaches surface generated by the arc as a limit;

that is, surface generated by the broken line approaches the zone AD as a limit;

volume generated by the polygon approaches volume generated by the sector;

that is, volume generated by the polygon approaches volume spherical sector AOD;

and OA is constant.

Therefore vol. sph. sect. AOD = area zone $AD \times \frac{1}{3}OA$. § 182

Q. E. D.

908. COR. I. Let H denote the altitude of the zone which forms the base of the spherical sector.

Then
$$\text{vol. sph. sector} = 2\pi RH \times \tfrac{1}{3}R$$
$$= \tfrac{2}{3}\pi R^2 H.$$

909. COR. II. *The volume of a sphere is equal to the area of its surface multiplied by one-third of its radius.*

Hint.—A sphere may be regarded as a spherical sector whose base is the surface of the sphere.

910. COR. III. *If V is the volume of a sphere, R its radius, and D its diameter,*
$$V = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3 = \tfrac{1}{6}\pi D^3.$$

911. COR. IV. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

PROBLEMS OF DEMONSTRATION

912. *Exercise.*—The lateral area of a cylinder of revolution is equal to the area of a circle the radius of which is a mean proportional between the altitude of the cylinder and the diameter of its base.

913. *Exercise.*—The volume of a cylinder is equal to the product of the area of a right section by an element.

914. *Exercise.*—The area of a sphere is equal to the lateral area of a circumscribed cylinder of revolution.

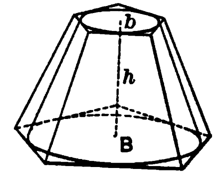
915. *Exercise.*—The volume of a sphere is two-thirds the volume of a circumscribed cylinder of revolution.

916. *Exercise.*—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of revolution of which the slant height is equal to the diameter of the base, be inscribed in a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

917. *Exercise.*—If a cylinder of revolution of which the altitude is equal to the diameter of the base, and a cone of revolution of which the slant height is equal to the diameter of the base, be circumscribed about a sphere; the total area of the cylinder is a mean proportional between the area of the sphere and the total area of the cone, and the volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

918. *Def.*—The **altitude** of a frustum of a cone is the perpendicular distance between its bases.

919. *Exercise.*—A frustum of a cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.



PROBLEMS FOR COMPUTATION

920. (1.) A right section of a cylinder is a circle whose radius is 3 ft.; an element of the cylinder is 13 ft. Find the lateral area.

(2.) A cylindrical boiler is 12 ft. long and 6 ft. in diameter. Find its surface, and the number of gallons of water it will hold.

(3.) A cylindrical pail is 6 in. deep and 7 in. in diameter. Find its contents and the amount of tin required for its construction.

(4.) A conical cistern is 13 in. deep and 12 in. across the top, which is circular. Find its contents.

(5.) A conical church steeple is 50 ft. high and 10 ft. in diameter at the base. How much would it cost to paint the steeple at 10 cents a square foot.

(6.) A cube, an edge of which is 2 in., is inscribed in a cone of revolution, of which the altitude is 5 in. Find the volume of the cone.

(7.) The sides of an equilateral triangle are each 10 in. What is the volume generated, if the triangle revolve about its altitude? What is the area of the surface generated?

(8.) The sides of a triangle are each 12.49 in. What is the volume generated if the triangle revolve about one side? What is the area of the surface generated?

(9.) Find the total area of a frustum of a cone of revolution, the radii of whose bases are 12 cm. and 7 cm., and whose altitude is 9.7 cm.

(10.) If the radius of a sphere is 647 cm.,

(*a*.) What is its area?

(*b*.) What is its volume?

(*c*.) What is the area of a lune whose angle is $35°$?

(11.) A sphere and a cylinder of revolution have equal areas. What is the ratio of the area of a sphere of half the diameter to the area of a similar cylinder of two-thirds the altitude?

(12.) How many marbles $\frac{3}{4}$ in. in diameter can be made from 100 cu. in. of glass, if there is no waste in melting?

(13.) Assuming the earth to be a sphere 7960 miles in diameter, what is the area of its surface? What is its volume?

(14.) The surface of a sphere is 1.514 sq. m. Find its radius.

(15.) The volume of a sphere is 1000 cu. in. Find its radius.

(16.) The surface of a sphere is 4632 sq. m. Find its volume.

(17.) Show that, if S is the surface of a sphere and V its volume,

$$36\pi V^2 = S^3.$$

(18.) A hollow rubber ball is 2 in. in diameter and the rubber is $\frac{3}{16}$ in. thick. How much rubber is used in the manufacture of 1000 such balls?

(19.) If a sphere of iron weighs 999 lbs., how much would a sphere of iron of one-third the diameter weigh?

(20.) A cone of revolution and a cylinder of revolution each have as base a great circle of a sphere, and as altitude the radius of the sphere. Find the ratios of the total surfaces of the cone and cylinder to the surface of the sphere.

(21.) Find the ratio of the volumes of a cone of revolution and a cylinder of revolution to the volume of a sphere, if the bases of the cone and cylinder are each equal to a great circle of the sphere, and the altitudes of the cone and cylinder are each equal to the diameter of the sphere.

(22.) If the radius of a sphere is 4.581 in., what is the area of a zone whose altitude is 1.456 in.?

(23.) A dome is in the form of a spherical zone of one base, and its height is 30 ft. Find its surface if the radius of the sphere is 35 ft.

(24.) The radius of a sphere is 6.742 in.; the altitude of a zone is 2 in. Find the volume of the spherical sector of which this zone is the base.

MISCELLANEOUS EXERCISES

IN

PLANE GEOMETRY

BOOK I

1. On a given line describe an isosceles triangle, each of whose equal sides shall be double the base.

2. The sum of the diagonals of a quadrilateral is less than the sum of any four lines that can be drawn from any point whatever (except the intersection of the diagonals) to the four angles.

3. Divide a right angle into three equal angles.

4. The bisector of an angle of a triangle is less than half the sum of the sides containing the angle.

5. In any right-angled triangle, the middle point of the hypotenuse is equally distant from the three angles.

6. If a straight line which bisects the vertical angle of a triangle also bisects the base, the remaining sides of the triangle are equal to each other.

7. If the base of an isosceles triangle be produced, the exterior angle exceeds one right angle by half the vertical angle.

8. The lines bisecting at right angles the sides of a triangle all meet in one point.

9. Construct an equilateral triangle, having given the length of the perpendicular drawn from one of the angles on the opposite side.

10. The difference between the acute angles of a right triangle is equal to the angle between the median and the perpendicular, drawn from the vertex of the right angle to the hypotenuse.

11. In a right triangle the bisector of the right angle also bisects the angle between the perpendicular and the median from the vertex of the right angle to the hypotenuse.

12. If, on the sides of a square, at equal distances from the four

angles, four points be taken, one on each side, the figure formed by joining those points will also be a square.

13. The parallelogram whose diagonals are equal is rectangular.

14. On a given line construct a square, of which the line shall be the diagonal.

15. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

16. Any line drawn through the centre of the diagonal of a parallelogram to meet the sides is bisected in that point.

17. The point of intersection of the straight lines which join the middle points of opposite sides of a quadrilateral is the middle point of the straight line joining the middle points of the diagonals.

18. The angle between the bisector of an angle of a triangle and the bisector of an exterior angle at another vertex is equal to half the third angle of the triangle.

19. Two quadrilaterals are equal if an angle of the one is equal to an angle of the other, and the four sides of the one are respectively equal to the four similarly situated sides of the other.

BOOK II

20. Draw a chord of given length through a given point, within or without a given circle.

21. Inscribe a circle in a given rhombus.

22. Any two chords of a circle which cut a diameter in the same point, and make equal angles with it, are equal to each other.

23. The two straight lines which join the opposite extremities of two parallel chords intersect in a point in that diameter which is perpendicular to the chords.

24. If two opposite sides of a quadrilateral inscribed in a circle are equal, the other two sides will be parallel.

25. All the equal chords in a circle may be touched by another circle.

26. Any number of triangles having the same base and the same vertical angle may be circumscribed by one circle.

27. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.

28. Describe a circle whose circumference shall pass through one angle and touch two sides of a given square.

29. If two circles be described, one circumscribed about and the other inscribed in a right-angled triangle, the sum of their diameters will be equal to the sum of the sides containing the right angle.

30. Construct a triangle, having given one angle, an adjacent side, and the sum of the other two sides.

31. Through a given point within a circle draw a chord which shall be bisected in that point.

32. Through a given point within a circle draw the least possible chord.

33. Two chords of a circle being given in magnitude and position, describe the circle.

34. Describe three equal circles touching one another; and also describe another circle which shall touch them all three.

35. The tangents to a circle at the extremities of any chord contain an angle which is twice the angle contained by the same chord and a diameter drawn from either of the extremities.

36. If from a point without a circle two tangents be drawn, the straight line which joins the point of contact will be bisected at right angles by a line drawn from the centre to the point without the circle.

37. If an arc of a circle be divided into three equal parts by three straight lines drawn from one extremity of the arc, the angle contained by two of the straight lines will be bisected by the third.

38. The sides of a right triangle are given in position, its hypotenuse in length. Find the locus of the middle point of the hypotenuse.

39. Divide a circle into two parts, such that the angle contained in one segment shall equal twice the angle contained in the other.

40. Describe a circle which shall touch a given circle in a given point, and also touch a given straight line.

41. With a given radius describe a circle which shall touch a given line and have its centre in another given line.

BOOK III

42. Upon a given base construct a right-angled triangle, having given the perpendicular from the right angle upon the hypotenuse.

43. Construct a triangle, having given one angle, a side opposite to it, and the sum of the other two sides.

44. In a given circle inscribe a triangle equiangular to a given triangle.

45. If a chord of a circular arc 16 in. in length be divided into two parts of 7 and 9 in. respectively by another chord, what is the length of the latter, one of its segments being 3 in.? *Ans.* 24 in.

46. If the chord of an arc be 720 ft., and the chord of its half be 369 ft., what is the diameter of the circle? *Ans.* 1681 ft.

47. If from a point without a circle two secants be drawn whose external segments are 8 in. and 7 in., while the internal segment of the latter is 17 in., what is the internal segment of the former?

Ans. 13 in.

48. If a circle be inscribed in a right-angled triangle, the sum of the two sides containing the right angle will exceed the hypotenuse by a line equal to the diameter of the inscribed circle.

49. Find a point in a given straight line such that tangents from it to two given circles shall be equal.

50. Construct a right triangle, having given one side and the perpendicular from the vertex of the right angle on the hypotenuse.

51. The distances from a point to the three nearest corners of a square are 1 in., 2 in., $2\frac{1}{2}$ in. Construct the square.

52. If the base of a triangle be 210, and the other sides 135 and 105, what is the length of the straight line drawn from the vertical angle to the point of bisection of the base? *Ans.* 60.

53. If two adjacent sides and one of the diagonals of a parallelogram be 245, 315, and 280, what is the length of the other diagonal?

Ans. 490.

54. If the base of a triangle be 54, and the other sides 75 and 48 respectively, what is the length of the external segment of the base made by a straight line bisecting the exterior angle at the vertex?

Ans. 96.

55. Two chords on opposite sides of the centre of a circle are parallel, and one of them has a length of 48, and the other of 14 in., the distance between them being 31 in.; what is the diameter of the circle? *Ans.* 50 in.

56. Two parallel chords on the same side of the centre of a circle whose diameter is 50 in. are measured, and found to be the one 48 and the other 14 in.; what is their distance apart? *Ans.* 17 in.

57. The sides of a right triangle are 15 ft. and 18 ft. The hypotenuse of a similar triangle is 20 ft. Find its sides.

58. The sides of a right triangle are 16.213 in. and 32.426 in. Find the ratio of the segments of the hypotenuse formed by the altitude upon the hypotenuse.

59. In an isosceles triangle the vertex angle is $45°$; each of the equal sides is 16 yds. Find the base in metres.

60. In a triangle whose sides are 247.93 mm., 641.98 mm., 521.23 mm., find the altitude upon the shortest side.

61. Construct a triangle, having given one angle, an adjacent side, and the sum of the other two sides.

62. Construct a triangle, having given one side, the angle opposite to it, and the ratio of the other two sides.

63. If the sides of a triangle are in the ratio of the numbers 2, 4, and 5, show whether it will be acute-angled or obtuse-angled.

64. If two similar fields together contain 518 square rods, what are their separate contents, their homologous sides being as 5 to 7?

Ans. 175 and 343 square rods.

65. The circle inscribed in an equilateral triangle has the same centre with the circle described about the same triangle, and the diameter of one is double that of the other.

66. If an equilateral triangle be inscribed in a circle, and the arcs cut off by two of its sides be bisected, the line joining the points of bisection will be trisected by the sides.

67. Given the hypotenuse of a right-angled triangle equal to 260 ft., and one of the legs equal to 224 ft., to find the other leg.

Ans. 132 ft.

68. Given the legs of a right-angled triangle equal to 765 and 408 yds. respectively, to compute the length of the perpendicular from the right angle to the hypotenuse. *Ans.* 360 yds.

69. If the sides of a triangle are 845, 910, and 975 respectively, what are the lengths of the segments into which they are severally divided by the perpendiculars from the opposite angles?

Ans. $\begin{cases} 350, \\ 495, \end{cases} \begin{cases} 325, \\ 585, \end{cases} \begin{cases} 429, \\ 546. \end{cases}$

70. Given the legs of a right-angled triangle equal to 455 and 1092 respectively, to compute the segments into which the hypotenuse is divided by the perpendicular from the right angle, and to compute also the perpendicular.

Ans. The segments are 175 and 1008, and the perpendicular 420.

71. Two adjacent sides of a parallelogram are 49 cm. and 53 cm. One diagonal is 58 cm. Find the other diagonal.

72. If an equilateral triangle be inscribed in a circle, each of its

sides will cut off one fourth part of the diameter drawn through the opposite angle.

BOOK IV

73. Trisect a given straight line, and hence divide an equilateral triangle into nine equal parts.

74. If the base and perpendicular of a triangle be 78 and 43 yds. respectively, what is the area? *Ans.* 1677 sq. yds.

75. If from any point in the diagonal of a parallelogram lines be drawn to the angles, the parallelogram will be divided into two pairs of equivalent triangles.

76. If the sides of any quadrilateral be bisected, and the points of bisection joined, the included figure will be a parallelogram and equal in area to half the original figure.

77. Bisect a triangle by a line drawn from a given point in one of the sides.

78. In any triangle, if a perpendicular be drawn from the vertex to the base, the difference of the squares upon the sides is equal to the difference of the squares upon the segments of the base.

79. The area of a rectangle is 18 sq. ft., and its base is 4.62 ft.; what is its altitude?

80. The base of one rectangle is 6 ft. and altitude 5 ft.; the base of another rectangle is 4 ft. and altitude 3 ft.; what is the ratio of the two rectangles?

81. Construct a square which shall have a given ratio to a given square.

82. Given the area and hypotenuse of a right-angled triangle to construct the triangle.

83. Prove, by a geometrical construction, that the square on the hypotenuse of a right triangle is equal to four times the triangle plus the square on the difference of the sides.

84. If from one of the acute angles of a right-angled triangle a straight line be drawn bisecting the opposite side, the square upon that line will be less than the square upon the hypotenuse by three times the square upon half the line bisected.

85. The square inscribed in a semicircle is to the square inscribed in the entire circle as 2 to 5.

86. The square inscribed in a semicircle is to the square inscribed in a quadrant of the same circle as 8 to 5.

87. A hexagon has its three pairs of opposite sides parallel. Prove that the two triangles which can be formed by joining alternate vertices are of equal area.

88. Equilateral triangles are constructed on the four sides of a square all lying within the square. Prove that the area of the star-shaped figure formed by joining the vertex of each triangle to the two nearest corners of the square is equal to eight times the area of one of the equilateral triangles minus three times the area of the square.

BOOK V

89. Trisect a given circle by dividing it into three equal sectors.

90. The centre of a circle being given, find two opposite points in the circumference by means of a pair of compasses only.

91. Divide a right angle into five equal parts.

92. Inscribe a square in a given segment of a circle.

93. Having given the difference between the diagonal and side of a square, describe the square.

94. Inscribe a square in a given quadrant.

95. Inscribe a circle in a given quadrant.

96. Describe a circle touching three given straight lines.

97. Within a given circle describe six equal circles touching each other and also the given circle, and show that the interior circle which touches them all is equal to each of them.

98. Within a given circle describe eight equal circles touching each other and the given circle.

99. The area of an equilateral triangle inscribed in a circle is equal to half that of the regular hexagon inscribed in the same circle.

100. The square of the side of an equilateral triangle inscribed in a circle is three times the square of the side of the regular hexagon inscribed in the same circle.

101. The area of a regular hexagon inscribed in a circle is three fourths of the regular hexagon circumscribed about the same circle.

102. What is the area of a circle whose diameter is 19?

103. What is the area of a circle whose circumference is 30?

104. What is the area of a quadrant of a circle whose radius is 11?

105. What is the diameter of a circle whose area is 40?

106. What is the circumference of a circle whose area is 35?

107. What is the circumference of the earth, supposing it to be a circle whose diameter is 7912 miles?

108. What is the circumference of a circle whose area is 27.45 square rods?

109. What is the area of a sector whose arc is one sixth of the circumference in a circle whose radius is 17 inches?

110. To construct a circumference whose length shall equal the sum of the lengths of two given circumferences.

111. To construct a circle equivalent to the sum of two given circles.

112. To inscribe a regular octagon in a given square.

113. To inscribe a regular hexagon in a given equilateral triangle.

114. Divide a given circle into any number of parts proportional to given straight lines by circumferences concentric with it.

115. An equilateral polygon inscribed in a circle is regular. An equilateral polygon circumscribed about a circle is regular, if the number of sides is odd.

116. An equiangular polygon inscribed in a circle is regular if the number of sides is odd. An equiangular polygon circumscribed about a circle is regular.

117. The diagonals of a regular pentagon are equal.

118. The pentagon formed by the diagonals of a regular pentagon is regular.

119. An inscribed regular octagon is equivalent to a rectangle whose sides are equal to the sides of an inscribed and a circumscribed square.

120. A man has a circular farm 640 acres in extent. He gives to each of his four sons one of the four largest equal circular farms which can be cut off from the original farm. How much did each son receive?

121. A man has a circular tract of land 700 acres in area; he wills one of the three largest equal circular tracts to each of his three sons, the tract at the centre included between the three circular tracts to his daughter, and the tracts included between the circumference of the original tract and the three circular tracts to his wife. How much will each receive?

MISCELLANEOUS EXERCISES

IN

SOLID GEOMETRY

BOOK VII

122. What is the entire surface of a triangular prism whose base is an equilateral triangle, having each of its sides equal to 17 in., and its altitude 5 ft.?

123. What is the entire surface of a regular triangular pyramid whose slant height is 15 ft., and each side of the base 4 ft.?

124. What is the convex surface of the frustum of a square pyramid whose slant height is 14 ft., each side of the lower base being $3\frac{1}{2}$ ft. and each side of the upper base $2\frac{1}{2}$ ft.?

125. What is the volume of a triangular prism whose height is 12 ft., and the three sides of its base 4, 5, and 6 ft.?

126. What is the volume of a triangular pyramid whose altitude is 25 ft., and each side of the base 4 ft.?

127. What is the volume of a piece of timber whose bases are squares, each side of the lower base being 14 in., and each side of the upper base 12 in., the altitude being 25 ft.?

128. The base of a rectangular parallelopiped is 3.42 ft. by 4.36 ft., and its volume is 100 cu. ft.; what is its altitude?

129. The volume of a parallelopiped is 366.4 cu. ft., and its altitude is 23.4 ft.; what is the area of its base?

130. The sides of the base of a tetraedron are 13, 15, and 17 ft., and its altitude is 11 ft.; required its volume.

131. What is the volume of a frustum of a regular triangular pyramid having a side of one base equal to 4 ft., and a side of the other base 3 ft., and the lateral edge equal to $3\frac{1}{2}$ ft.?

132. Find the lateral area, total area, and volume of a regular

triangular prism the perimeter of whose base is 16.413 in. and whose altitude is 14.718 in.

133. Find the lateral area, total area, and volume of a regular hexagonal pyramid each side of whose base is 8.84 in. and whose altitude is 4.92 in.

134. The area of the base of a pyramid is 13 sq. m.; its altitude is 4 m. Find the area of a section parallel to the base and distant $1\frac{1}{2}$ m. from it. Also find the volume of the pyramid cut off by this plane.

135. Find the volume of a frustum of a pyramid whose base is a regular octagon having each side equal to 4 in., and whose altitude is 9 in., made by a plane 5 in. from the vertex.

136. The diagonal of a cube is 24.16 cm. Find its surface and volume.

137. The volume of a polyedron is 984.62 cu. ft. Find the volume of a similar polyedron whose edges are nine times the edges of the first polyedron.

BOOK VIII

138. The area of the base of a circular cone is 43 sq. in. Its altitude is 19 in. Find the area of a section parallel to the base and 10 in. from the vertex.

139. If the area of a circle of a sphere distant 10 cm. from its centre is 40 sq. cm., find the radius of the sphere.

140. The polar distance of a circle of a sphere is 30°. If its circumference is 6 m., what is the radius of the sphere?

141. Find the area in square feet of a lune whose angle is 36° on a sphere whose surface is 46 sq. m.

142. If the area of a spherical triangle whose angles are 110°, 46°, and 150°, is 84.662 sq. yds., what is the area of a trirectangular triangle on the same sphere?

143. The angles of a spherical pentagon are 68°, 97°, 156°, 80°, and 142°. Its area is 8 sq. ft. Find the area of the sphere.

BOOK IX

144. What is the entire surface of a cylinder whose altitude is 17 ft., and the diameter of its base 3 ft.?

145. What is the entire surface of a cone whose side is 24 ft., and the diameter of its base 5 ft.?

146. What is the entire surface of a frustum of a cone whose side is 18 ft., and the radii of the bases 5 ft. and 4 ft.?

147. What is the volume of a cylinder whose altitude is 16 ft., and the circumference of its base 5 ft.?

148. What is the volume of a cone whose altitude is 13 ft. and the circumference of whose base is 7 ft.?

149. What is the volume of a frustum of a cone of which the altitude is 22 ft., the circumference of its lower base 18 ft., and that of the upper base 14 ft.?

150. If the perimeter of a right section of a cylinder is 16 in. and its lateral area is 256 sq. in., what is the length of an element?

151. Find the volume of a cylinder of revolution whose total area is 160π and whose radius is 4.

152. Find the radius of a cylinder of revolution whose total area is 80π and whose altitude is 6.

153. An oil tank is in the form of a circular cylinder. If the tank is 26 ft. long and 78 in. in diameter, how many liters of oil will it contain?

154. Find the volume of a cone of revolution whose total area is 200π and whose altitude is 16.

155. The lateral area of a cone of revolution is 39π. Its altitude is 9. Find the height of an equivalent cylinder of revolution whose radius is 4.

156. What is the surface of a sphere, the circumference of its great circle being 40 ft.?

157. What is the area of the surface of the earth, supposing it to be a sphere whose diameter is 7912 miles?

158. What is the convex surface of a zone whose altitude is 13 in. upon a sphere whose diameter is 40 in.?

159. What is the volume of a sphere whose diameter is 17 in.?

160. What is the volume of the earth, supposing it to be a sphere whose diameter is 7912 miles?

161. What is the volume of a spherical sector, the diameter of the sphere being 12 ft. and the altitude of the zone which forms its base being 3 ft.?

162. The volume of a sphere is 1870 cubic feet; required its radius.

163. The edge of a cube is 30 inches; required the volume of the circumscribing sphere.

164. The element of a right cone is 22 ft. and its altitude 15 ft.; required its lateral surface.

165. A stone obelisk has the form of a regular quadrangular pyramid, having a side of its base equal to 4 ft. and its slant height 13 ft. The density of the stone is 2.5 times that of water. What is its weight, assuming that a cubic foot of water weighs $62\frac{1}{2}$ pounds?

166. Supposing the earth to be a sphere, and that a quadrant is equal to 32,800,000 ft., it is required to determine the radius of the earth, the area of its surface, its volume, and its weight, the mean density of the earth being 4.5 times that of water, the weight of a cubic foot of water being $62\frac{1}{2}$ lbs.

167. In a sphere whose diameter is 14 in. the altitude of a zone of one base is 2 in. Find the altitude of a cylinder of revolution whose lateral area shall equal the area of the zone and whose base shall equal the base of the zone.

168. Find the radius of a circle whose area shall equal the area of a zone of altitude 16.954 m. on a sphere whose diameter is 20 m.

169. Find the radius of a sphere whose area shall equal the area of the zone in the previous example.

170. A conical glass is 5 in. high and 4 in. across at the top. A marble is within the glass, and water is poured in till the marble is just immersed. If the amount of water poured in is $\frac{1}{2}$ the contents of the glass, what is the diameter of the marble.

171. If two spheres of radii 13 in. and 8 in. are inscribed in a cone of revolution so that the greater may touch the less and also the base of the cone, find the volume of the cone.

TABLE OF MEASURES AND WEIGHTS

English Measures

LENGTH

12 inches (in.)	=1 foot (ft.).
3 feet	=1 yard (yd.).
5½ yards	=1 rod (rd.).
4 rods	=1 chain (ch.).
80 chains	=1 mile (m.).
1 yard	= .9144 meter.
1 mile	=1.6093 kilometers.

SURFACE

144 sq. inches	=1 sq. foot.
9 sq. feet	=1 sq. yard.
30¼ sq. yards	=1 sq. rod.
160 sq. rods	=1 acre.
640 acres	=1 sq. mile.
1 sq. yard	=0.8361 sq. meter.
1 acre	=0.4047 hectare.

VOLUME

1728 cu. inches	=1 cu. foot.
27 cu. feet	=1 cu. yard.
128 cu. feet	=1 cord (cd.).
1 cu. yard	=0.7646 cu. meter.
1 cord	=3.625 steres.

ANGLES

60 seconds (″)	=1 minute (′).
60 minutes	=1 degree (°).
90 degrees	=1 right angle.

CIRCLES

360 degrees=1 circumference.

$\pi=3.1416=$ nearly $3^1/_7$.

CAPACITY

1 liq. gal.=3.785	liters=	231 cu. in.
1 dry gal.=4.404	liters=	268.8 cu. in.
1 bushel =0.3524	hkl.	=2150.42 cu. in.

AVOIRDUPOIS WEIGHT

16 ounces (oz.).	=1 pound (lb.).
100 lbs.	=1 hundredweight (cwt.).
20 hundredweight	=1 ton (T.).

1 pound=.4536 kilo.=7000 grains.
1 ton =.9071 tonneau.

Metric Measures

LENGTH

10 millimeters (mm.).	=1 centimeter (cm.).
10 centimeters	=1 decimeter (dcm.).
10 decimeters	=1 meter (m.).
10 meters	=1 dekameter (dkm.).
10 dekameters	=1 hektometer (hkm.).
10 hektometers	=1 kilometer (km.).
1 meter	=39.37 inches
1 kilometer	= 0.6214 mile.

SURFACE

100 sq. millimeters	=1 sq. centimeter.
100 sq. centimeters	=1 sq. decimeter.
100 sq. decimeters	= { 1 sq. meter. / 1 centare (ca.).
100 centares	=1 are (a.).
100 ares	=1 hektare (hka.).
1 sq. centimeter	=0.1550 sq. inch.
1 sq. meter	=1.196 sq. yards.
1 are	=3.954 sq. rods.
1 hektare	=2.471 acres.

VOLUME

1000 cu. millimeters	=1 cu. centimeter.
1000 cu. centimeters	=1 cu. decimeter.
1000 cu. decimeters	=1 cu. meter.
	=1 stere (st.).
1 cu. centimeter	=0.061 cu. inch.
1 cu. meter	=1.308 cu. yards.
1 stere	=0.2759 cord.

CAPACITY

100 centiliters (cl.)	=1 liter (l.).
100 liters	=1 hektoliter (hkl.).

1 liter=1.0567 liq. qts.=1 cu. dcm.

METRIC WEIGHT

1000 grams (gm.)	=1 kilogram (kilo.).
1000 kilograms	=1 tonneau (t.).
1 gram	=15.432 grains.
1 kilogram	= 2.2046 pounds.
1 tonneau	= 1.1023 tons.

INDEX OF DEFINITIONS

[The references are to sections.]

INDEX OF CONSTRUCTIONS IN PLANE GEOMETRY

INDEX OF RULES OF MENSURATION